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The eigenvalues of compact self-adjoint operators on Hilbert spaces

Mitra Bandari
San Jose State University

THE EIGENVALUES OF COMPACT SELF-ADJOINT OPERATORS ON HILBERT SPACES

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Mitra Bandari

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APPROVED FOR THE DEPARTMENT OF MATHEMATICS

Dr. Timothy Hsu

Maria Cayco

Dr. Maria Cayco

Samih Obaid

Dr. Samih Obaid

APPROVED FOR THE UNIVERSITY

Pamela C Stacks

ABSTRACT

THE EIGENVALUES OF COMPACT SELF-ADJOINT OPERATORS ON HILBERT SPACES

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This thesis illustrates the proof of the Hilbert-Schmidt theorem, which states: If $A : H \to H$ is a compact self-adjoint operator on a Hilbert space H, then there exists an orthonormal set $B = \{e_n\}_{n=1}^{\infty} \subseteq H$ such that for all n, e_n is an eigenvector of A with eigenvalue λ_n. Furthermore, B is a basis for H, $\lim_{n\to\infty} \lambda_n = 0$, and the dimension of the eigenspace of A associated with the nonzero eigenvalue λ_n is finite.

After reviewing the Lebesgue integral of measurable functions, and defining various function spaces such as $L^p(\mathbb{R}^n)$ and $\ell_2(\mathbb{Z}^+)$, we develop the basic properties of inner product spaces. Next, we prove that function spaces such as $\ell_2(\mathbb{Z}^+)$ and $L^2(\mathbb{R})$ are Hilbert spaces.

After discussing the geometry of Hilbert spaces, we obtain the Riesz-Fischer theorem, which states: finite dimensional and infinite dimensional Hilbert spaces are isomorphic to $\mathbb{C}^n$, and $\ell_2(\mathbb{Z}^+)$ respectively.

Finally, we prove the Hilbert-Schmidt theorem.

DEDICATION

To my family and to my professors at San Jose State University.

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Thanks to my wonderful family who have supported me throughout my educational journey.

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CHAPTER 1

INTRODUCTION

1.1 Why study the eigenvalues of compact self-adjoint operators?

This thesis illustrates the proof of the Hilbert-Schmidt theorem, which states: If $A : H \to H$ is a compact self-adjoint operator on a Hilbert space H, then there exists an orthonormal set $B = \{e_n\}_{n=1}^{\infty} \subseteq H$ such that for all n, e_n is an eigenvector of A with eigenvalue λ_n. Furthermore, B is a basis for H, $\lim_{n\to\infty} \lambda_n = 0$, and for any nonzero eigenvalue λ_n, the dimension of the eigenspace of A associated with λ_n is finite (i.e., each nonzero λ_n has finite multiplicity).

The eigenvalues λ_n of compact self-adjoint operators on Hilbert spaces H are interesting because they determine solutions to many problems in applied mathematics. For instance, in a uniform vibrating string with length L, we want to solve the one dimensional wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{1.1}$$

with a suitable initial condition and the boundary conditions:

$$u(0,t) = 0, u(L,t) = 0.$$

First, assume a special product solution of the form $u(x,t) = \phi(x)h(t)$. Then it turns out (see Haberman [4, p. 143]) for some $\lambda \in \mathbb{R}$,

$$\frac{1}{h}\left(\frac{\partial^2 h}{\partial t^2}\right) = -\lambda c^2 \tag{1.2}$$

and

$$\frac{1}{\phi(x)}\left(\frac{\partial^2 \phi(x)}{\partial x^2}\right) = -\lambda \tag{1.3}$$

are solutions to equation (1.1).

Furthermore, it can be shown that the operators $\frac{\partial^2}{\partial t^2}$ in ((1.2)) or $\frac{\partial^2}{\partial x^2}$ in ((1.3)) with eigenvalues $-\lambda c^2$ and $-\lambda$ respectively have an inverse T, which is a compact self-adjoint operator. So each eigenvalue μ of T (see Reed and Simon [6, p. 370-371]) provides an eigenvalue $\lambda = \frac{1}{\mu}$ for the operators $\frac{\partial^2}{\partial t^2}$ and $\frac{\partial^2}{\partial x^2}$.

In a two dimensional case, a vibrating membrane fixed along its boundary D has a wave equation:

$$\frac{\partial^2 u(x,y,t)}{\partial t^2} = c^2 \left(\frac{\partial^2 u(x,y,t)}{\partial x^2} + \frac{\partial^2 u(x,y,t)}{\partial y^2} \right) \tag{1.4}$$

with a suitable initial condition and a homogeneous boundary condition $u = 0$ along the entire boundary D.

First, assume the special product solution of the form $u(x,y,t) = \phi(x,y)h(t)$. Then it turns out (see Haberman [4, p. 276-277]) for some $\lambda \in \mathbb{R}^2$,

$$\frac{1}{h}\left(\frac{\partial^2 h}{\partial t^2}\right) = -\lambda c^2 \tag{1.5}$$

and

$$\frac{1}{\phi}\left(\frac{\partial^2 \phi(x,y)}{\partial x^2} + \frac{\partial^2 \phi(x,y)}{\partial y^2}\right) = -\lambda \tag{1.6}$$

are solutions to equation (1.4).

In addition, it can be shown that the operators

$$\frac{\partial^2}{\partial t^2} \text{ in } ((1.5))$$

or

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \text{ in } ((1.6)),$$

with eigenvalue$-\lambda c^2$ and $-\lambda$ respectively, have an inverse T, a compact self-adjoint operator. Therefore, each eigenvalue ν of T (see Reed and Simon [6, p. 370-371]) provides an eigenvalue $\lambda = \frac{1}{\nu}$ for the operators $\frac{\partial^2}{\partial t^2}$ and ∇^2.

1.2 Overview

In Chapter 2, we introduce measurable functions, integrable by Lebesgue integral, and we investigate the properties, definitions and theorems that lead to the study of the geometry of various function spaces such as $L^p(\mathbb{R}^n)$ and $\ell_2(\mathbb{Z}^+)$.

In Chapter 3, we discuss the foundation of Hilbert spaces by defining inner products on $\mathbb{C}^n$, a complex valued function space, and $L^2(X)$, where X is a measurable subset of $\mathbb{R}^n$. Furthermore, we define isomorphism of inner product spaces to classify Hilbert spaces.

In Chapter 4, we continue to build our definition of Hilbert spaces, providing various examples. In addition, we complete the motivation behind the uniqueness of Hilbert spaces by the exposition of the proof of following theorem: Any Hilbert space H is the span of a countable orthonormal set $B \subseteq H$, a set of eigenvectors of a compact self-adjoint operator A on H.

Chapter 5 provides definitions of bounded subsets of inner product spaces and bounded linear functions on Hilbert spaces. Furthermore, we construct other Hilbert spaces such as $L^2(X)$ by using the the following theorem: Any closed subspace of a Hilbert space is a Hilbert space.

Chapter 6 covers an exposition of the proof of Hilbert-Schmidt theorem, which helps to find solutions to wave equations in applied mathematics.

CHAPTER 2

LEBESGUE MEASURE AND LEBESGUE INTEGRAL

2.1 Measurable functions

Definition 2.1.1. (Reed and Simon [6, p. 12]). A collection M of subsets of $\mathbb{R}^n$ is said to be a σ-algebra (sigma algebra) in $\mathbb{R}^n$ if

(1) The empty set $\emptyset$ belongs to M.

(2) If $A \in M$ then $\mathbb{R}^n \backslash A$ belongs to M.

(3) If $\{A_n\}$ is any sequence of elements in M, then a countable union of these elements, $\bigcup_{n=1}^{\infty} A_n$, belongs to M.

Definition 2.1.2. (Reed and Simon [6, p. 14]). The Borel sets, $\mathcal{B}_n$, are the smallest collection of subsets in $\mathbb{R}^n$ with the following properties:

(1) $\mathcal{B}_n$ contains all open subsets of $\mathbb{R}^n$.

(2) $\mathcal{B}_n$ is a σ-algebra.

Theorem 2.1.3. *(Reed and Simon [6, p. 16]). There exists a σ-algebra $\mathcal{M}_n$ containing the Borel sets $\mathcal{B}_n$ and a function $\mu : \mathcal{M}_n \to [0, \infty]$ such that:*

(1) If a nonempty set $A \in \mathcal{M}_n$ has measure zero (i.e., $\mu(A) = 0$), then $\mathcal{M}_n$ contains every subset of A. Also, the empty set $\emptyset$ has measure zero $(\mu(\emptyset) = 0)$.

(2) For any $E = [a_1, b_1] \times \ldots \times [a_n, b_n]$, $\mu(E) = (b_1 - a_1) \times \ldots \times (b_n - a_n)$.

(3) For $E_n \in \mathcal{M}_n$, if $E_1, E_2, \ldots$ are disjoint, then $\mu(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i)$; otherwise, $\mu(\cup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} \mu(E_i)$.

(4) (Royden [7, P. 62].) Let $E_n \in \mathcal{M}_n$ be an infinite decreasing sequence of sets, i.e., a sequence with $E_{n+1} \subset E_n$ for all $n \in \mathbb{N}$. If $\mu(E_1) < \infty$, then

$$\mu\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} \mu(E_n). \tag{2.1}$$

(5) *If* $E_n \in \mathcal{M}_n$ *and* $y \in \mathbb{R}^n$, *then* $E + y = \{x + y | x \in E\} \in \mathcal{M}_n$.

(6) $\mu(E + y) = \mu(E)$ *for* $E \in \mathcal{M}_n$ *and* $y \in \mathbb{R}^n$.

(7) *Let* $E_n \in \mathcal{M}_n$, *and let* A *be an* $n \times n$ *orthogonal matrix (that means* $AA^T = I$*). Then* $E \in \mathcal{M}_n$ *and* $\mu(AE) = \mu(E)$.

Definition 2.1.4. The sets in the collection $\mathcal{M}_n$ are called the measurable subsets of $\mathbb{R}^n$. For $E \in \mathcal{M}_n$, $\mu(E)$ is called the measure of E.

Definition 2.1.5. A real-valued function $f : \mathbb{R}^n \to \mathbb{R}$ is measurable if for every $\alpha \in \mathbb{R}$,

$$\{x \in \mathbb{R}^n | f(x) > \alpha\} \tag{2.2}$$

is measurable (is an element of $\mathcal{M}_n$).

Definition 2.1.6. Equivalence classes of functions: Two measurable complex-valued functions f and g are identified almost everywhere (a.e.) if

$$\{x \in \mathbb{R}^n | f(x) \neq g(x)\} \tag{2.3}$$

has measure zero.

Definition 2.1.7. Let $f(x) = u(x) + iv(x)$ be a complex-valued function such that $f : \mathbb{R}^n \to \mathbb{C}$. Then $f(x)$ is called a complex-valued measurable function if $u : \mathbb{R}^n \to \mathbb{R}$ and $v : \mathbb{R}^n \to \mathbb{R}$ are measurable.

Definition 2.1.8. Let $f : \mathbb{R}^n \to \mathbb{C}$. Then $f^+ = \max\{f(x), 0\}$ and $f^- = -\min\{f(x), 0\}$.

Theorem 2.1.9. *Let* $\alpha \in \mathbb{C}$, *and let the functions* $f : X \to \mathbb{C}$ *and* $g : X \to \mathbb{C}$ *be measurable complex-value functions on* X, *a measurable subset of* $\mathbb{R}^n$. *Then* $f + g$, αf *and* fg *are measurable. Also, let* $\{f_n\}$ *be a sequence of measurable functions on* X. *Then* $\sup\{f_n\}$, $\inf\{f_n\}$, f^+, *and* f^- *are measurable.*

Proof. See Royden [7, p. 67]. □

Definition 2.1.10. (Royden [7, p. 70].) If E is a measurable set in $\mathbb{R}^n$, then the characteristic function χ_E is defined on $\mathbb{R}^n$ by

$$\chi_E = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

A linear combination of χ_{E_i} provides a real valued function $\phi = \sum_{i=1}^n a_i \chi_{E_i}$ which is said to be a positive simple function if the a_i are all positive.

Lemma 2.1.11. *(Royden [7, p. 65]). Let* E *be a set of finite measure. For every* $\epsilon > 0$, *there exists a finite union of open intervals* U *such that if*

$$S = \{x \mid x \in E \cap x \notin U\} \cup \{x \mid x \notin E \cap x \in U\}, \tag{2.4}$$

then $\mu(S) < \epsilon$.

Proof. See Royden [7, p. 65]. □

2.2 Lebesgue integral of measurable functions

Definition 2.2.1. (Royden [7, p. 60].) If $\phi = \sum_{i=1}^{n} a_i \chi_{E_i}$ is a positive simple function, then

$$\int_{\mathbb{R}^n} \phi = \sum_{i=1}^{n} a_i \mu(E_i)$$

is said to be the Lebesgue integral of ϕ.

Definition 2.2.2. (Royden [7, p. 60].) If $f : \mathbb{R}^n \to [0, \infty]$ is a measurable function, then

$$\int_{\mathbb{R}^n} f = \sup_{0 \leq s \leq f, s \text{ simple}} \int_{\mathbb{R}^n} s$$

is called the Lebesgue integral of f.

Definition 2.2.3. A measurable function $f : \mathbb{R}^n \to [0, \infty]$ is summable if $\int_{\mathbb{R}^n} f$ is finite. (Note that $\int_{\mathbb{R}^n} f$ always exists, but it may be infinity.)

Definition 2.2.4. (Royden [7, p. 90].) Let a real-valued function $f : \mathbb{R}^n \to \mathbb{R}$ be measurable, let $f^+(x) = \max\{f(x), 0\}$, and let $f^-(x) = -\min\{f(x), 0\}$. If f^+ and f^- are both summable, then

$$\int_{\mathbb{R}^n} f = \int_{\mathbb{R}^n} f^+ - \int_{\mathbb{R}^n} f^-$$

is defined to be the Lebesgue integral of f, and we say that f is summable.

Definition 2.2.5. Let $f = u(x) + iv(x)$ be a measurable function such that $f : \mathbb{R}^n \to \mathbb{C}$. If $u(x)$ and $v(x)$ are both integrable, then

$$\int_{\mathbb{R}^n} f = \int_{\mathbb{R}^n} u + i \int_{\mathbb{R}^n} v$$

is said to be the Lebesgue integral of f, and we say that f is summable.

Definition 2.2.6. Let E be a measurable set in $\mathbb{R}^n$, and let

$$\chi_E = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

be the characteristic function of E. If f is measurable, we define

$$\int_E f = \int_{\mathbb{R}^n} f \chi_E.$$

Theorem 2.2.7. *Let X be a measurable subset in $\mathbb{R}^n$. Then $f : X \to \mathbb{C}$ is summable on X if and only if $|f|$ is summable on X.*

Proof. Let X be a measurable subset in $\mathbb{R}^n$. Then the first case is when $f : X \to \mathbb{R}$. In that case $f = f_+ - f_-$ and $|f| = f_+ + f_-$ are both summable if and only if f_+ and f_- are summable. In the second case, let $f(x) = u(x) + iv(x)$. If f is summable, then u and v are summable, so $|u|$ and $|v|$ are summable. Then since

$$|f(x)| = |u(x) + iv(x)| \leq |u(x)| + |v(x)|, \tag{2.5}$$

$|f(x)|$ is summable.

Conversely, if $|f|$ is summable, then since $|u(x)| \leq |f(x)|$ and $|v(x)| \leq |f(x)|$, $|u|$ and $|v|$ are summable. This implies that u and v are summable, so f is summable. □

Theorem 2.2.8. *Assume on a measurable subset X in $\mathbb{R}^n$, that the measurable functions $f : X \to \mathbb{C}$ and $g : X \to \mathbb{C}$ are summable.*

(1) If $\alpha \in \mathbb{C}$, then $(\alpha f + g)$ is summable, and $\int_X (\alpha f + g) = \int_X \alpha f + \int_X g$.

(2) If f and g are real valued functions on X, and $f \leq g$, then $\int_X f \leq \int_X g$. In particular, if $g \geq 0$ then $\int_X g \geq 0$.

(3) When A and B are disjoint subsets of X, then $\int_{A \cup B} f = \int_A f + \int_B f$.

(4) $\left|\int_X f\right| \leq \int_X |f|$.

Proof. See Royden [7, p. 90]. □

Lemma 2.2.9. *Let X be a measurable subset in $\mathbb{R}^n$, and let $f : X \to [0, \infty]$ be a nonnegative measurable function on X. If $\int_X f = 0$, then $f = 0$ almost everywhere (a.e.)*

Proof. Let $f : X \to [0, \infty]$, $\int_X f = 0$, and $E = \{x | f(x) > 0\}$. Now, let

$$E_k = \{x \in X \mid f(x) \geq \frac{1}{k}\} \subseteq E, \tag{2.6}$$

for all $k \in \mathbb{N}$. We have $E = \cup_{k=1}^{\infty} E_k$, and $\{E_k\}_{k=1}^{\infty}$ are measurable subsets of E. So $\mu(E) \leq \sum_{k=1}^{\infty} \mu(E_k)$. Now, if $\mu(E_k) = \epsilon > 0$, since $f \geq \frac{1}{k}\chi_{E_k} > 0$, we have

$$\int_X f \geq \frac{1}{k} \int_X \chi_{E_k} = \frac{1}{k}\mu(E_k) > 0.$$

This means $\int_X f \neq 0$, which contradicts the hypothesis. Thus, $\mu(E_k) = 0$ for all k. Hence, $\mu(E) = 0$. □

Theorem 2.2.10. *(Royden [7, p. 86].) Fatou's lemma: Let E be a measurable subset of $\mathbb{R}^n$. If $\{f_n\}_{n=1}^{\infty}$ is a sequence of nonnegative measurable functions such that $f_n \to f$ almost everywhere on E, then*

$$\int_E f \leq \liminf \int_E f_n.$$

Proof. See Royden [7, p. 86]. □

Theorem 2.2.11. *(Royden [7, p. 91].) The Lebesgue Convergence theorem: Let X be a measurable subset of $\mathbb{R}^n$, and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions $f_n : X \to \mathbb{C}$. If g is a nonnegative real-valued summable function on X such that $|f_n| \leq g$, and for almost all $x \in X$, $\lim_{n\to\infty} f_n(x) = f(x)$, then*

$$\int_X f = \lim \int_X f_n.$$

Proof. See Royden [7, p. 91]. □

2.3 L^p spaces

Definition 2.3.1. If X is a subset of $\mathbb{R}^n$, then the space of all possible complex-valued functions defined on X is $F(X) = \{f \mid f : X \to \mathbb{C}\}$.

Theorem 2.3.2. *$F(X)$ is a vector space.*

Proof. See Messer [5, p. 33]. □

Definition 2.3.3. Let X be a measurable subset in $\mathbb{R}^n$, let $0 < p < \infty$, and let $f : X \to \mathbb{C}$ be measurable. If $|f|^p$ is summable, then we define

$$\|f\|_p = \left(\int_X |f|^p\right)^{\frac{1}{p}}. \tag{2.7}$$

We define $L^p(X)$ to be the set of all measurable functions $f : X \to \mathbb{C}$ such that $|f|^p$ is summable (i.e., $\|f\|_p < \infty$).

Theorem 2.3.4. *The Minkowski inequality: If $f : X \to \mathbb{C}$ and $g : X \to \mathbb{C}$ are in $L^p(X)$ for $1 \leq p < \infty$, then*

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p. \tag{2.8}$$

Proof. See Reed and Simon [6, p. 350]. □

Theorem 2.3.5. *$L^p(X)$ is a subspace of $F(X)$.*

Proof. If f and $g \in L^p(X)$ and $\alpha \in \mathbb{C}^n$, then

(1) $0 \in L^p(X)$ because $\|0\|_p = 0 < \infty$.

(2) $\|f\|_p < \infty$, and $\|g\|_p < \infty$, so $\|f\|_p + \|g\|_p < \infty$. Now, by theorem 2.3.4,

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p < \infty, \tag{2.9}$$

so $f + g \in L^p(X)$.

(3) Since $\|f\|_p < \infty$,

$$\|\alpha f\|_p^p = \int_p |\alpha f|^p = |\alpha|^p \int_x |f|^p = |\alpha|^p \|f\|_p < \infty. \tag{2.10}$$

Thus, $\|\alpha f\|_p < \infty$.

$L^p(X)$ is closed under addition, complex constant multiplication, and it contains the zero function. Therefore, $L^p(X)$ is a subspace of $F(X)$. □

2.4 The space $\ell_2(\mathbb{Z}^+)$

Definition 2.4.1. We define $\ell_2(\mathbb{Z}^+)$ to be the elements of the set of all complex sequences $y = (y(1), y(2), y(3), \ldots)$ that satisfy

$$\sum_{k=1}^{\infty} |y(k)|^2 < \infty. \tag{2.11}$$

As with $L^p(X)$, it can be shown $\ell_2(\mathbb{Z}^+)$ is a subspace of $F(\mathbb{Z}^+)$.

Theorem 2.4.2. *(Fatou's lemma for $\ell_2(\mathbb{Z}^+)$): (Reed and Simon [6, p. 24].) Suppose $y_n(k)$ is a sequence in $\ell_2(\mathbb{Z}^+)$. If $y_n(k)$ is nonnegative for all $n, k \geq 1$ and $\lim_{n\to\infty} y_n(k) = w(k)$ where $w(k) \in \mathbb{R}$, then $\sum_{k=1}^{\infty} w(k) \leq \liminf_{n\to\infty} \sum_{k=1}^{\infty} y_n(k)$.*

Proof. See Reed and Simon ([6, p. 24] theorem 1.17). □

Theorem 2.4.3. *(Lebesgue Convergence Theorem for $\ell_2(\mathbb{Z}^+)$): (Reed and Simon [6, p. 24].) Suppose $y_n(k)$ is a sequence in $\ell_2(\mathbb{Z}^+)$. If $y_n(k)$ is nonnegative for all $n, k \geq 1$, and $\lim_{n\to\infty} y_n(k) = w(k)$ where $w(k) \in \mathbb{R}$, and there exists a nonnegative sequence $v(k)$ such that $\sum_{k=1}^{\infty} v(k) < \infty$ and $|y_n(k)| \leq v(k)$, then $\sum_{k=1}^{\infty} w(k) = lim_{n\to\infty} \sum_{k=1}^{\infty} y_n(k)$.*

Proof. See Reed and Simon ([6, p. 24] theorem 1.16). □

2.5 Metric spaces

Definition 2.5.1. A metric on a set V is a function $d : V \times V \to \mathbb{R}$ such that for every $x, y, z \in V$,

(1) $d(x, y) \geq 0$, with $d(x, y) = 0$ if and only if $x = y$.

(2) $d(x, y) = d(y, x)$.

(3) Triangle inequality: $d(x, y) \leq d(x, z) + d(y, z)$.

A metric space is a set V equipped with a metric d on V.

Definition 2.5.2. Let V be a metric space, and let S be a subset in V. If for all $f \in V$ and $\epsilon > 0$, there exists $g \in S$ such that $d(f,g) < \epsilon$, then S is dense in V.

Definition 2.5.3. A metric space V is separable if there exists a countable dense subset S in V.

Definition 2.5.4. A sequence $\{x_n\}_{n=1}^{\infty}$ in a metric space V is Cauchy if for any $\epsilon > 0$ there exists an N such that for any $n > m > N$ we have $d(x_n, x_m) < \epsilon$.

Definition 2.5.5. We say a sequence $\{x_n\}$ in a metric space V converges to $x \in V$ when for any $\epsilon > 0$ there exists N such that if $n > N$, then $d(x_n, x) < \epsilon$.

Theorem 2.5.6. *In a metric space V, every convergent sequence is Cauchy.*

Proof. (Royden [7, p. 124].) Let V be a metric space, and $\{x_n\}_1^{\infty}$ be a sequence in V that converges to $x \in V$. Given $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ $|x_n - x| < \frac{\epsilon}{2}$. Also, for all $m > n \geq N$ we have $|x - x_m| < \frac{\epsilon}{2}$. Therefore,

$$|x_n - x_m| \leq |x_n - x| + |x - x_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

□

Definition 2.5.7. Let V be a metric space. If every Cauchy sequence in V converges to an element in V, then (V,d) is complete.

Theorem 2.5.8. *$\mathbb{R}$ is complete.*

Proof. See Bartle and Sherbert[1, p. 34]. □

Theorem 2.5.9. *$\mathbb{Q}$ is countable and dense in $\mathbb{R}$.*

Proof. See Bartle and Sherbert [1, p. 19]. □

Theorem 2.5.10. *$\mathbb{R}$ is separable.*

Proof. By theorem 2.5.9 and definition 2.5.3, $\mathbb{Q}$ is countable and dense in $\mathbb{R}$, so $\mathbb{R}$ is separable. □

CHAPTER 3

INNER PRODUCT SPACES

3.1 Inner product spaces

Definition 3.1.1. (Messer [5, p. 136].) Let V be a complex vector space. We say that $\langle\ ,\ \rangle : V \times V \to \mathbb{C}$ is an inner product on V if for every $u, v, w \in V$ and $\alpha, \beta \in \mathbb{C}$ we have

(1) $\langle \alpha u + \beta v, w\rangle = \alpha\langle u, w\rangle + \beta\langle v, w\rangle$.

(2) $\langle u, v\rangle = \overline{\langle v, u\rangle}$.

(3) $\langle u, u\rangle \geq 0$, and $\langle u, u\rangle = 0$ if and only if $u = 0$.

By the second property $\langle u, u\rangle = \overline{\langle u, u\rangle}$; this means $\langle u, u\rangle$ is a real number because real numbers are the only complex numbers that are equal to their conjugates.

The length or norm of $u \in V$ is given by $\|u\| = \sqrt{\langle u, u\rangle}$. Also note

$$\langle u, \alpha v + \beta w\rangle = \langle u, \alpha v\rangle + \langle u, \beta w\rangle = \overline{\alpha}\langle u, v\rangle + \overline{\beta}\langle u, w\rangle. \tag{3.1}$$

Example 3.1.2. Let $\langle\ ,\ \rangle$ be an inner product on V, and W be a subspace of V. Now, every element in W belongs to V, so for all u, v, w in W and $a, b \in \mathbb{C}$,

(1) $\langle \alpha u + \beta v, w\rangle = \alpha\langle u, w\rangle + \beta\langle v, w\rangle$.

(2) $\langle u, v\rangle = \overline{\langle v, u\rangle}$.

(3) $\langle u, u\rangle \geq 0$ and $\langle u, u\rangle = 0$ if and only if $u = 0$.

So $\langle\ ,\ \rangle$ is an inner product on W.

Example 3.1.3. Let $\mathbb{C}^n$ be the standard n-dimensional vector space over $\mathbb{C}$. Then $\langle u, v\rangle = u^T\overline{v}$ is defined for all $u, v \in \mathbb{C}^n$.

Theorem 3.1.4. *$\langle u, v\rangle = u^T\overline{v}$ is an inner product on $\mathbb{C}^n$.*

Proof. Suppose $\langle u, v\rangle = u^T\overline{v}$. Now for all

$$u = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, \; v = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}, \; w = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{C}^n$$

and α, $\beta \in \mathbb{C}$, we have

(1) $\langle \alpha u + \beta v, w\rangle = (\alpha u + \beta v)^T\overline{w} = \alpha u^T\overline{w} + \beta v^T\overline{w} = \alpha\langle u, w\rangle + \beta\langle v, w\rangle$.

(2) $\langle u, v\rangle = u^T\overline{v} = \left(u^T\overline{v}\right)^T = \overline{v}^T u = \overline{v^T\overline{u}} = \overline{\langle v, u\rangle}$ since $u^T\overline{v}$ is a 1×1 matrix.

(3) $\langle u, u\rangle = u^T\overline{u} \geq 0$ because $u^T\overline{u} = |a_1|^2 + \ldots + |a_n|^2 \geq 0$.
In addition, $\langle u, u\rangle = u^T\overline{u} = 0$ if and only if $a_1 = \ldots = a_n = 0$.

Hence, $\langle u, v\rangle = u^T\overline{v}$ is an inner product. □

Lemma 3.1.5. *If $y(k), z(k) \in \ell_2(\mathbb{Z}^+)$, then $\sum_{n=1}^{\infty} y(k)\overline{z}(k)$ is finite.*

Proof. Note that

$$0 \leq \left(|y(k)| - \left|\overline{z(k)}\right|\right)^2 = |y(k)|^2 - 2\,|y(k)|\left|\overline{z(k)}\right| + \left|\overline{z(k)}\right|^2. \tag{3.2}$$

However, $\left|\overline{z(k)}\right|^2 = |z(k)|^2$. Hence,

$$2\,|y(k)|\left|\overline{z(k)}\right| \leq |y(k)|^2 + |z(k)|^2. \tag{3.3}$$

Therefore,

$$\left|y(k)\overline{z(k)}\right| \leq \frac{1}{2}\,|y(k)|^2 + \frac{1}{2}\,|z(k)|^2. \tag{3.4}$$

Then

$$\sum_{k=1}^{\infty}\left|y(k)\overline{z(k)}\right| \leq \frac{1}{2}\sum_{k=1}^{\infty}|y(k)|^2 + \frac{1}{2}\sum_{k=1}^{\infty}|z(k)|^2. \tag{3.5}$$

The left hand side of the inequality is a nonnegative series bounded above by right hand side, so $\sum_{k=1}^{\infty} y(k)\overline{z(k)}$ converges absolutely. □

Definition 3.1.6. For all $y(k), z(k) \in \ell_2\left(\mathbb{Z}^+\right)$ we define

$$\langle y(k), z(k)\rangle = \sum_{k=1}^{\infty} y(k)\overline{z(k)}. \tag{3.6}$$

This is well defined by Lemma 3.1.5.

Theorem 3.1.7. *$\langle y(k), z(k)\rangle$ is an inner product on $\ell^2(\mathbb{Z}^+)$.*

Compare with theorem 3.1.10.

Proof. Suppose $\langle y(k), z(k)\rangle = \sum_{k=1}^{\infty} y(k)\overline{z}(k)$. Then for every $w(k), y(k), z(k) \in \ell^2(\mathbb{Z}^+)$ and $\alpha, \beta \in \mathbb{C}$ we have:

(1)

$$\begin{aligned}\langle(\alpha w(k) + \beta y(k)), z(k)\rangle &= \sum_{k=1}^{\infty}(\alpha w(k) + \beta y(k))\overline{z}(k) \\ &= \sum_{k=1}^{\infty} \alpha w(k)\overline{z}(k) + \sum_{k=1}^{\infty} \beta y(k)\overline{z}(k) \\ &= \alpha\langle w(k), z(k)\rangle + \beta\langle y(k), z(k)\rangle.\end{aligned} \tag{3.7}$$

(2) $\langle y(k), z(k)\rangle = \sum_{k=1}^{\infty} y(k)\overline{z}(k) = \sum_{k=1}^{\infty} \overline{(z(k)\overline{y}(k))} = \overline{\langle z(k), y(k)\rangle}$.

(3) Note that $\langle y(k), y(k)\rangle = \sum_{k=1}^{\infty} |y(k)|^2 \geq 0$ because $|y(k)|^2 \geq 0$.

(4) If $y(k) = 0$, then $\|y(k)\|_2^2 = \sum_{k=1}^{\infty} |y(k)|^2 = 0$. Conversely, if some $y(m) \neq 0$, then $\sum_{k=1}^{\infty} |y(k)|^2 \geq |y(m)|^2 > 0$.

Therefore, $\langle y(k), z(k)\rangle$ is an inner product on $\ell^2(\mathbb{Z}^+)$. □

Lemma 3.1.8. *Let X be a measurable subset of $\mathbb{R}^n$. If $f, g \in L^2(X)$, then $f\overline{g}$ is summable. This means*

$$\int_X f\overline{g} \tag{3.8}$$

converges.

Proof. We have

$$0 \leq \left(|f(x)| - \left|\overline{g(x)}\right|\right)^2 = |f(x)|^2 - 2\,|f(x)|\left|\overline{g(x)}\right| + \left|\overline{g(x)}\right|^2. \tag{3.9}$$

However, $\left|\overline{g(x)}\right|^2 = |g(x)|^2$. Hence,

$$2\,|f(x)|\left|\overline{g(x)}\right| \leq |f(x)|^2 + |g(x)|^2. \tag{3.10}$$

Therefore,

$$\left|f(x)\overline{g(x)}\right| \leq \frac{1}{2}\,|f(x)|^2 + \frac{1}{2}\,|g(x)|^2. \tag{3.11}$$

Then

$$\int_X |f\overline{g}| \leq \frac{1}{2}\int_X |f|^2 + \frac{1}{2}\int_X |g|^2. \tag{3.12}$$

By definition 2.3.3, $|f|^2$ and $|g|^2$ are summable. So by the theorem 2.2.7, $\int_X f\overline{g}$ converges. □

Definition 3.1.9. On a measurable subset X of $\mathbb{R}^n$, we define

$$\langle f, g\rangle = \int_X f\bar{g}, \tag{3.13}$$

which is well-defined for $f, g \in L^2(X)$ by Lemma 3.1.8.

Theorem 3.1.10. *For a measurable subset X in $\mathbb{R}^n$, the function $\langle f, g\rangle$ is an inner product on $L^2(X)$.*

Proof. Suppose $\langle f, g\rangle = \int_X f\bar{g}$. Then for every $f, g, h \in X$ and $\alpha, \beta \in \mathbb{C}$ we have:

(1) $\langle(\alpha f + \beta h), g\rangle = \int_X (\alpha f + \beta h)\overline{g} = \int_X \alpha f\overline{g} + \int_X \beta h\overline{g} = \alpha\langle f, g\rangle + \beta\langle h, g\rangle$.

(2) $\langle f, g\rangle = \int_X f\overline{g} = \int_X \overline{(g\overline{f})} = \overline{\langle g, f\rangle}$.

(3) By theorem 2.2.8 (2), $\langle f, f\rangle = \|f\|^2 = \int_X |f|^2 \geq 0$ because $|f|^2 \geq 0$.

(4) If $\langle f, f\rangle = \|f\|^2 = \int_X |f|^2 = 0$, then by lemma 2.2.9, $f = 0$. Moreover, if $f = 0$, then $\|f\|^2 = 0$. Thus, $\langle f, f\rangle = \|f\|^2 = 0$ if and only if $f = 0$.

The proof is complete. □

3.2 Isomorphisms of inner product spaces

Definition 3.2.1. Let V and W be inner product spaces, and let $T : V \to W$ be a linear function on V. Then T is an isomorphism (of inner product spaces) if

(1) T is a linear bijection.

(2) For all $v_1, v_2 \in V$, $\langle Tv_1, Tv_2\rangle_W = \langle v_1, v_2\rangle_V$.

Lemma 3.2.2. *Let V and W be inner product spaces, and let $T : V \to W$ be a linear bijection. Then T is an isomorphism if and only if $\|Tv\|_W = \|v\|_V$ for any $v \in V$.*

Proof. $1 \Rightarrow 2$. Let $T : V \to W$ be an isomorphism. Then by definition 3.2.1, for any $v \in V$, $\|Tv\|_W^2 = \langle Tv, Tv\rangle_W = \langle v, v\rangle_V = \|v\|_V^2$.

$2 \Rightarrow 1$. Consider

$$\begin{aligned}
\|T(v_1 + v_2)\|_W^2 &= \|Tv_1 + Tv_2\|_W^2 \\
&= \langle Tv_1 + Tv_2, Tv_1 + Tv_2\rangle_W \\
&= \|Tv_1\|_W^2 + \langle Tv_2, Tv_1\rangle_W + \langle Tv_1, Tv_2\rangle_W + \|Tv_2\|_W^2 \\
&= \|Tv_1\|_W^2 + 2\,\mathrm{Re}\,\langle Tv_1, Tv_2\rangle_W + \|Tv_2\|_W^2 .
\end{aligned} \tag{3.14}$$

Also,

$$\begin{aligned}
\|v_1 + v_2\|_V^2 &= \langle v_1 + v_2, v_1 + v_2\rangle_V \\
&= \|v_1\|_V^2 + 2\,\mathrm{Re}\,\langle v_1, v_2\rangle_V + \|v_2\|_V^2 .
\end{aligned} \tag{3.15}$$

Since $\|Tv_1\|_W^2 = \|v_1\|_V^2$ and $\|Tv_2\|_W^2 = \|v_2\|_V^2$, then $2\,\mathrm{Re}\,\langle Tv_1, Tv_2\rangle_W = 2\,\mathrm{Re}\,\langle v_1, v_2\rangle_V$. Similarly, from $\|v_1 + iv_2\|_V^2 = \|T(v_1 + iv_2)\|_W^2$, it follows that $\mathrm{Im}\,\langle Tv_2, Tv_1\rangle_W = \mathrm{Im}\,\langle v_2, v_1\rangle_V$. Therefore, by definition 3.2.1, T is an isomorphism. □

3.3 The Schwarz inequality

Theorem 3.3.1. *Let V be a vector space, and let $\langle\ ,\ \rangle$ be an inner product on V. Then for every u and $v \in V$,*

$$|\langle u, v\rangle| \leq \|u\|\|v\|. \qquad \textit{(Schwarz inequality)} \tag{3.16}$$

Proof. (Griffel [3, p. 177].) If $v = 0$, then $\langle u, 0\rangle = 0$, so the Schwarz inequality holds. Now, suppose $v \neq 0$ and $\alpha \in \mathbb{C}$. Then

$$\begin{aligned} 0 \leq \|u - \alpha v\|^2 &= \langle u - \alpha v, u - \alpha v\rangle \\ &= \langle u, u\rangle - \overline{\alpha}\langle u, v\rangle - \alpha[\langle v, u\rangle - \overline{\alpha}\langle v, v\rangle]. \end{aligned} \tag{3.17}$$

By definition 3.1.1

$$\langle u, u\rangle = \|u\|^2 \text{ and } \langle v, v\rangle = \|v\|^2, \tag{3.18}$$

so

$$0 \leq \|u - \alpha v\|^2 = \|u\|^2 - \overline{\alpha}\langle u, v\rangle - \alpha[\langle v, u\rangle - \overline{\alpha}\|v\|^2]. \tag{3.19}$$

Let

$$\alpha = \frac{\langle u, v\rangle}{\langle v, v\rangle} = \frac{\langle u, v\rangle}{\|v\|^2} \text{ and } \overline{\alpha} = \frac{\overline{\langle u, v\rangle}}{\|v\|^2} = \frac{\langle v, u\rangle}{\|v\|^2}. \tag{3.20}$$

By inserting (3.20) in (3.19) we have:

$$0 \leq \|u - \alpha v\|^2 = \|u\|^2 - \frac{\overline{\langle u, v\rangle}}{\|v\|^2}\langle u, v\rangle - \alpha\left[\langle v, u\rangle - \frac{\langle v, u\rangle}{\|v\|^2}\|v\|^2\right]. \tag{3.21}$$

Then

$$0 \leq \|u\|^2 - \frac{|\langle u, v\rangle|^2}{\|v\|^2} \tag{3.22}$$

$$|\langle u, v\rangle|^2 \leq \|u\|^2\|v\|^2. \tag{3.23}$$

□

Corollary 3.3.2. *For every $y(k), z(k) \in \ell_2(\mathbb{Z}^+)$,*

$$|\langle y(k), z(k)\rangle| \leq \|y(k)\|\|z(k)\| \tag{3.24}$$

Proof. Follows from the Cauchy-Schwarz inequality (Theorem 3.3.1) since $\langle\ ,\ \rangle$ is an inner product on $\ell_2(\mathbb{Z}^+)$. □

Corollary 3.3.3. *(Messer [5, p. 149].) Cauchy-Schwarz inequality: If X is a measurable set in $\mathbb{R}^n$, then for every f and $g \in L^2(X)$ we have*

$$\left|\int_X f\overline{g}\right|^2 \leq \int_X |f|^2 dx \int_X |g|^2 dx. \tag{3.25}$$

Proof. Follows from the Cauchy-Schwarz inequality (Theorem 3.3.1) because $\langle\ ,\ \rangle$ is an inner product on $L^2(X)$. □

Corollary 3.3.4. *The Triangle Inequality: In an inner product space V, for every $f, g \in V$,*

$$\|f+g\| \leq \|f\| + \|g\|. \tag{3.26}$$

Proof. (Griffel [3, p. 149].) By the definition of inner product (Definition 3.1.1),

$$\begin{aligned}\|f+g\|^2 &= \langle f+g, f+g\rangle \\ &= \langle f,f\rangle + \langle f,g\rangle + \overline{\langle f,g\rangle} + \langle g,g\rangle \\ &\leq \langle f,f\rangle + 2\left|\langle f,g\rangle\right| + \langle g,g\rangle.\end{aligned} \tag{3.27}$$

By Cauchy-Schwarz,

$$\begin{aligned}\langle f,f\rangle + 2\left|\langle f,g\rangle\right| + \langle g,g\rangle &\leq \langle f,f\rangle + 2\sqrt{\langle f,f\rangle\langle g,g\rangle} + \langle g,g\rangle \\ &= \left[\sqrt{\langle f,f\rangle} + \sqrt{\langle g,g\rangle}\right]^2.\end{aligned} \tag{3.28}$$

By definition 3.1.1,

$$\langle f,f\rangle = \|f\|^2 \text{ and } \langle g,g\rangle = \|g\|^2. \tag{3.29}$$

Thus,

$$\|f+g\| \leq \|f\| + \|g\|. \tag{3.30}$$

□

3.4 Inner product spaces are metric spaces

Definition 3.4.1. Suppose x, y belong to an inner product space V. Then the norm metric on V is

$$d(x,y) = \sqrt{\left|\langle x-y, x-y\rangle\right|} = \|x-y\|. \tag{3.31}$$

The following theorem shows that this is a metric (Definition 2.5.1) on V.

Theorem 3.4.2. *Every inner product space is a metric space (with the norm metric).*

Proof. Suppose x, y, z belong to an inner product space V. Now, by definition 3.4.1,

$$\|x-y\| = 0 \iff \langle x-y, x-y\rangle = 0,$$

so by definition 3.1.1, $\langle x-y, x-y\rangle = 0 \iff x-y = 0$. Thus, $x = y \iff d(x,y) = 0$. In particular, $d(x,x) = \|x-x\| = 0$.

(1) $d(x,y) = \|x-y\| = \|y-x\| = d(y,x)$.

(2) $\sqrt{\left|\langle x-y, x-y\rangle\right|} = \|x-y\|$, and by Corollary 3.3.4,

$$d(x,y) = \|x-y\| = \|x-z+z-y\| \leq \|x-z\| + \|z-y\|,$$

so

$$d(x,y) \leq \|x-z\| + \|z-y\| = \sqrt{\left|\langle x-z, x-z\rangle\right|} + \sqrt{\left|\langle z-y, z-y\rangle\right|} = d(x,z) + d(z,y).$$

Thus, an inner product space V is a metric space with respect to the norm metric. □

3.5 Infinite series in inner product spaces

Definition 3.5.1. (Bartle and Sherbert[1, p. 89]). Let $\{v_n\}_{n=1}^{\infty}$ be a sequence in an inner product space V, and let $s = \sum_{n=1}^{\infty} v_n$ be an infinite series with partial sums $s_k = \sum_{n=1}^{k} v_n$. If $\lim_{k\to\infty} s_k = v$ for some $v \in V$, then s converges to v. Otherwise, s diverges. Furthermore, the sum and difference of any two convergent series converges.

Theorem 3.5.2. *(Bartel and Sherbert[1, p. 90]). Let $s = \sum_{n=1}^{\infty} v_n$ be an infinite series in an inner product space V. The series $s = \sum_{n=1}^{\infty} v_n$ converges in V if and only if for every $\epsilon > 0$ there exists an $M \in \mathbb{N}$ such that if $n > m > M$, then*

$$\left| \sum_{k=m-1}^{n} v_k \right| = |s_n - s_m| < \epsilon.$$

Proof. See Bartel and Sherbert[1, p. 90]. □

3.6 Orthonormal sets and Bessel's inequality

Definition 3.6.1. (Orthonormal sets: Dym and McKean [2, p. 22]). Let S be a subset of an inner product space V. We say that S is orthonormal if every $v \in S$ has length one ($\|v\| = 1$) and for all $v \neq w$ in S, $\langle v, w \rangle = 0$.

Lemma 3.6.2. *If $\{e_i\}_{i=1}^{n}$ is a finite orthonormal set in an inner product space V and $c_i, d_i \in \mathbb{C}$, then*

$$\left\langle \sum_{i=1}^{n} c_i e_i, \sum_{i=1}^{n} d_i e_i \right\rangle = \sum_{i=1}^{n} c_i \overline{d_i}. \tag{3.32}$$

Proof. By definition of inner product (Definition 3.1.1), for all $i, j \in \mathbb{N}$, we have

$$\left\langle \sum_{i=1}^{n} c_i e_i, \sum_{j=1}^{n} d_j e_j \right\rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} c_i \overline{d_j} \langle e_i, e_j \rangle. \tag{3.33}$$

If $i = j$, then $c_i \overline{d_j} \langle e_i, e_j \rangle = c_i \overline{d_j}$; otherwise $c_i \overline{d_j} \langle e_i, e_j \rangle = 0$. □

Theorem 3.6.3. *(Bessel's inequality: Dym and McKean [2, p. 25].) Let $\{e_j\}_{j=1}^{\infty}$ be an orthonormal set in an inner product space V, and for any $v \in V$ and $n \in \mathbb{Z}^+$, let $\langle v, e_n \rangle = c_n$. Then*

$$\sum_{n=1}^{\infty} |c_n|^2 \leq \|v\|^2. \tag{3.34}$$

Proof. Observe that

$$
\begin{aligned}
0 &\leq \left\| v - \sum_{k=1}^{n} \langle v, e_k \rangle e_k \right\|^2 \\
&= \left\langle v - \sum_{k=1}^{n} \langle v, e_k \rangle e_k, v - \sum_{k=1}^{n} \langle v, e_j \rangle e_j \right\rangle \\
&= \langle v, v \rangle - \sum_{k=1}^{n} \langle v, e_k \rangle \langle e_k, v \rangle - \sum_{j=1}^{n} \overline{\langle v, e_j \rangle} \langle v, e_j \rangle + \sum_{k=1}^{n} \langle v, e_k \rangle \overline{\langle v, e_k \rangle} \text{ by lemma 3.6.2} \\
&= \langle v, v \rangle - 2 \sum_{k=1}^{n} |\langle v, e_k \rangle|^2 + \sum_{k=1}^{n} |\langle v, e_k \rangle|^2 \\
&= \langle v, v \rangle - \sum_{k=1}^{n} |\langle v, e_k \rangle|^2 \\
&= \|v\|^2 - \sum_{k=1}^{n} |c_k|^2 .
\end{aligned}
\tag{3.35}
$$

Thus, $0 \leq \|v\|^2 - \sum_{k=1}^{n} |c_k|^2$ implies

$$
\sum_{k=1}^{n} |c_k|^2 \leq \|v\|^2, \tag{3.36}
$$

which is a Bessel's inequality for a finite set. For an infinite orthonormal set $\{e_n\}_{n=1}^{\infty} \subseteq V$, we observe that $\sum_{n=1}^{\infty} |c_n|^2$ converges to a real number $c \leq \|v\|$ since it is increasing and bounded above by $\|v\|$. $\square$

CHAPTER 4

HILBERT SPACES

4.1 $\mathbb{C}^n$ is a Hilbert space

Definition 4.1.1. An inner product space V is called a Hilbert space if V is a complete (Definition 2.5.7) and separable (Definition 2.5.3) metric space. (See theorem 3.4.2).

Lemma 4.1.2. *$\mathbb{C}$ is complete.*

Proof. Let $\{x_k + y_k i\}_{k=1}^{\infty}$ be a Cauchy sequence in $\mathbb{C}$. This means for $\epsilon > 0$ we may choose an N such that if $k > j > N$ then

$$d(x_k + y_k i, x_j + y_j i) = \sqrt{(x_k - x_j)^2 + (y_k - y_j)^2} < \epsilon.$$

Then

$$d(x_k, x_j) = |x_k - x_j| = \sqrt{(x_k - x_j)^2} \leq \sqrt{(x_k - x_j)^2 + (y_k - y_j)^2} < \epsilon.$$

Thus, $d(x_k, x_j) < \epsilon$. Hence $\{x_k\}_{k=1}^{\infty}$ is Cauchy in $\mathbb{R}$, and similarly $\{y_k\}_{k=1}^{\infty}$ is Cauchy in $\mathbb{R}$ because

$$d(y_k, y_j) = |y_k, y_j| = \sqrt{(y_k - y_j)^2} \leq \sqrt{(x_k - x_j)^2 + (y_k - y_j)^2} < \epsilon.$$

By the theorem 2.5.8 and definition 2.5.7, $\{x_k\}_{k=1}^{\infty}$ converges to a real number x, and $\{y_k\}_{k=1}^{\infty}$ also converges to a real number y. Now for $\epsilon > 0$, choose an N such that for $k > N$, we have $|x_k - x| < \frac{\epsilon}{2}$, $|y_k - y| < \frac{\epsilon}{2}$. Then

$$d(x_k + y_k i, x + yi) = \sqrt{(x_k - x)^2 + (y_k - y)^2} \leq |x_k - x| + |y_k - y| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \quad (4.1)$$

Therefore, there exists $x + yi \in \mathbb{C}$ such that $x_k + y_k i \to x + yi$ as $k \to \infty$. □

Lemma 4.1.3. *$\mathbb{C}^n$ is complete.*

Proof. Let $\{z_j\}_{j=1}^{\infty} = \{z_{j1}, z_{j2}, \ldots, z_{jn}\}$ be a sequence in $\mathbb{C}^n$. We first show that for a fixed ℓ, $\{z_{j\ell}\}_{j=1}^{\infty}$ is a Cauchy sequence. We know that for given $\epsilon > 0$ there exists M such that for all $j, k \in \mathbb{N}$, if $j > k > M$, then $|z_j - z_k| < \epsilon$. Hence

$$d\left(z_j, z_k\right) = \sqrt{\left|z_{j1} - z_{k1}\right|^2 + \ldots + \left|z_{j\ell} - z_{k\ell}\right|^2 + \ldots + \left|z_{jn} - z_{kn}\right|^2} < \epsilon. \tag{4.2}$$

Now in this case, we know that

$$\begin{aligned} d\left(z_{j\ell}, z_{k\ell}\right) &= \sqrt{\left|z_{j\ell} - z_{k\ell}\right|^2} \\ &\leq \sqrt{\left|z_{j1} - z_{k1}\right|^2 + \ldots + \left|z_{j\ell} - z_{k\ell}\right|^2 + \ldots + \left|z_{jn} - z_{kn}\right|^2} \\ &< \epsilon. \end{aligned} \tag{4.3}$$

Because $\{z_{j\ell}\}_{j=1}^{\infty} \subseteq \mathbb{C}$ is Cauchy and $\mathbb{C}$ is complete, $z_{j\ell} \to w_\ell$ for some $w_\ell \in \mathbb{C}$.

Now, let $w = (w_1, w_2, \ldots, w_n)$, and choose N such that if $j > N$, then $|z_{j\ell} - w_\ell| < \frac{\epsilon}{n}$. For $j > N$, we know that $d\left(z_{j\ell}, w_\ell\right) = |z_{j\ell} - w_\ell| < \frac{\epsilon}{n}$, so

$$\begin{aligned} d\left(z_j, w\right) &= \sqrt{\left|z_1 - w_1\right|^2 + \left|z_2 - w_2\right|^2 + \ldots + \left|z_n - w_n\right|^2} \\ &\leq \left|z_1 - w_1\right| + \left|z_2 - w_2\right| + \ldots + \left|z_n - w_n\right| \\ &< \frac{\epsilon}{n} + \ldots + \frac{\epsilon}{n} = \epsilon. \end{aligned} \tag{4.4}$$

Therefore, $z_j \to w$ as $j \to \infty$. □

Lemma 4.1.4. *$\mathbb{C}$ is separable; specifically, the rational complex numbers $\mathbb{Q}(i)$ are dense in $\mathbb{C}$.*

Proof. Let $z = x + yi \in \mathbb{C}$, and let $\mathbb{Q}(i) = \{q = r + si \mid r, s \in \mathbb{Q}\}$ be the set of rational complex numbers. If a and $b \in \mathbb{R}$ with $a < b$ then there exists a rational number $r \in \mathbb{Q}$ such that $a < r < b$. For $\epsilon > 0$, choose a rational number r such that $x - \frac{\epsilon}{2} < r < x$, and choose a rational $s \in \mathbb{Q}$ such that $y - \frac{\epsilon}{2} < s < y$. If $q = r + si$, then

$$d(z, q) = \sqrt{|x - r|^2 + |y - s|^2} \leq |x - r| + |y - s| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \tag{4.5}$$

Thus, the countable set $\mathbb{Q}(i)$ is dense in $\mathbb{C}$, so $\mathbb{C}$ is separable by definition 2.5.3. □

Lemma 4.1.5. *$\mathbb{C}^n$ is separable; specifically, $\mathbb{Q}(i)^n$ is dense in $\mathbb{C}^n$.*

Proof. Since $\mathbb{Q}(i)$ is dense in $\mathbb{C}$, for $\epsilon > 0$ and $(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$, we choose rational complex numbers $r_j \in \mathbb{Q}(i)$ such that $|z_j - r_j| < \frac{\epsilon}{n}$. Also, let $r = (r_1, r_2, \ldots, r_n) \in \mathbb{Q}(i)^n$. Then we have

$$\begin{aligned} d\left(z, r\right) &= \sqrt{\left|z_1 - r_1\right|^2 + \ldots + \left|z_n - r_n\right|^2} \\ &\leq \left|z_1 - r_1\right| + \left|z_2 - r_2\right| + \ldots + \left|z_n - r_n\right| \\ &< \frac{\epsilon}{n} + \ldots + \frac{\epsilon}{n} = \epsilon. \end{aligned} \tag{4.6}$$

$\mathbb{C}^n$ is separable by definition 2.5.3 since the countable set $\mathbb{Q}(i)^n$ is dense in $\mathbb{C}^n$. □

Theorem 4.1.6. *$\mathbb{C}^n$ is a Hilbert space.*

Proof. See lemma 4.1.3 and lemma 4.1.5. □

4.2 Absolute summability

Definition 4.2.1. (Royden [7, p. 124].) Let V be an inner product space. A series $\sum_{i=1}^{\infty} f_n$ with $f_n \in V$ is called absolutely summable if $\sum_{i=1}^{\infty} \|f_n\| < \infty$.

Lemma 4.2.2. *Let V be an inner product space. If every absolutely summable series converges in V, then V is a complete metric space.*

Proof. (Royden [7, p. 125].) Suppose every absolutely summable series converges in V, and let $\{f_n\}_{n=1}^{\infty}$ be a Cauchy sequence in V. So for $\epsilon > 0$, there exists an $N(\epsilon) \in \mathbb{N}$ such that if $\ell > k > N(\epsilon)$ then $\|f_\ell - f_k\| < \epsilon$. Now, we choose

$$N_1 = N(2^{-1}) + 1 \tag{4.7}$$

and

$$N_k = \max\left(N(2^{-k}) + 1, N_{k-1} + 1\right). \tag{4.8}$$

Next, let $g_1 = f_{N_1}, g_2 = f_{N_2} - f_{N_1}, \ldots, g_\ell = f_{N_\ell} - f_{N_{\ell-1}}$. We have

$$\sum_{k=1}^{\infty} \|g_k\| = \|g_1\| + \sum_{k=2}^{\infty} \|g_k\|. \tag{4.9}$$

For $k \geq 2$,

$$\|g_k\| = \|f_{N_k} - f_{N_{k-1}}\| < 2^{-(k-1)}, \tag{4.10}$$

since $N_{k-1} > N\left(2^{-(k-1)}\right)$. So

$$\begin{aligned}\sum_{k=1}^{\infty} \|g_k\| &= \|g_1\| + \sum_{k=2}^{\infty} \|g_k\| \\ &< \|g_1\| + \sum_{k=2}^{\infty} 2^{-(k-1)} \\ &= \|g_1\| + 1 < \infty.\end{aligned} \tag{4.11}$$

So $\sum_{k=1}^{\infty} g_k$ converges to some $f \in V$ because $\sum_{k=1}^{\infty} g_k$ is absolutely summable.

Note that

$$\sum_{k=1}^{\ell} g_k = f_{N_1} + f_{N_2} - f_{N_1} + f_{N_3} - f_{N_2} + \ldots + f_{N_\ell} - f_{N_{\ell-1}} = f_{N_\ell}.$$

Thus, $f_{N_\ell} \to f$ as $\ell \to \infty$. Now, we show that $f_n \to f$ as $n \to \infty$. For $\epsilon > 0$, there exists $M_1 \in \mathbb{N}$ such that if $\ell > M_1$ then $\|f_{N_\ell} - f\| < \frac{\epsilon}{2}$, and there exists $M_2 \in \mathbb{N}$ such that if $m, n > M_2$, then $\|f_n - f_m\| < \frac{\epsilon}{2}$. Let $M = \max\ \{M_1, M_2\}$. For $n > M$, choose $k > n > M$, and note that $N_k \geq k > n > M$. Hence,

$$\begin{aligned} \|f_n - f\| &\leq \|f_n - f_{N_k}\| + \|f_{N_k} - f\| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned} \tag{4.12}$$

Thus, $f_n \to f$. Therefore, by definition 2.5.7, V is complete. □

4.3 $\ell_2(\mathbb{Z}^+)$ is a Hilbert space

Lemma 4.3.1. *Every absolutely summable series in $\ell_2(\mathbb{Z}^+)$ converges (in the metric space $\ell_2(\mathbb{Z}^+)$).*

Proof. Let y_n be a sequence in $\ell_2(\mathbb{Z}^+)$ such that $\sum_{n=1}^{\infty} \|y_n\|_2 = M < \infty$. For fixed k, let $u_n(k) = \sum_{m=1}^{n} |y_m(k)|$ be the sequence of partial sums of $\sum_{n=1}^{\infty} |y_n(k)|$. Then by the triangle inequality (Corollary 3.3.4),

$$\|u_n\|_2 \leq \sum_{m=1}^{n} \|y_m\|_2 \leq M. \tag{4.13}$$

So $\sum_{k=1}^{\infty} [u_n(k)]^2 \leq M^2$. Since $u_n(k)$ is a nonnegative and increasing function of n, then $\lim_{n\to\infty} u_n(k) = u(k)$, which is either a nonnegative real number or ∞. However, by Fatou's lemma (Theorem 2.2.10),

$$\sum_{k=1}^{\infty} [u(k)]^2 \leq \liminf_{n\to\infty} \sum_{k=1}^{\infty} [u_n(k)]^2 \leq M^2. \tag{4.14}$$

This means $u(k) \in \ell^2(\mathbb{Z}^+)$. So $u(k) < \infty$ for all $k \in \mathbb{Z}^+$ and the series $\sum_{n=1}^{\infty} y_n(k)$ converges absolutely to a complex number $s(k)$ for all $k \in \mathbb{Z}^+$. Note that $|s(k)| \leq u(k)$ and $\sum_{m=1}^{n} |y_m(k)| \leq u(k)$. Thus, we have

$$|y_m(k) - s(k)| \leq |y_m(k)| + |s(k)| \leq 2u(k). \tag{4.15}$$

By the Lebesgue Convergence Theorem (theorem 2.4.3), $s \in \ell_2(\mathbb{Z}^+)$ and $\|y_n - s\|_2 \to 0$. Therefore, every absolutely summable series $\sum_{n=1}^{\infty} |y_n|$ in $\ell^2(\mathbb{Z}^+)$ converges to some $s \in \ell_2(\mathbb{Z}^+)$. □

Lemma 4.3.2. *$\ell_2(\mathbb{Z}^+)$ is separable. Specifically, let $S = \{$ all $r \in \mathbb{Q}(i)^\infty \mid$ there exists an $N \in \mathbb{N}$ such that $r(k) = 0$ for $k > N\}$. Then S is dense in $\ell_2(\mathbb{Z}^+)$.*

Proof. Fix $\epsilon > 0$ and $y = (y(1), y(2), \ldots) \in \ell_2(\mathbb{Z}^+)$. Since $\sum_{k=1}^{\infty} |y(k)|^2$ converges, and $\mathbb{Q}_i$ is dense in $\mathbb{C}$, choose $N \in \mathbb{N}$ such that $\sum_{k=N+1}^{\infty} |y(k)|^2 < \left(\frac{\epsilon}{2}\right)^2$, and pick $r(1), \ldots, r(N) \in \mathbb{Q}(i)$ such that $|y(k) - r(k)| < \left(\frac{\epsilon}{2^k}\right)\left(\frac{1}{2}\right)$. So if

$$r = (r(1), \ldots, r(N), 0, 0, \ldots), \tag{4.16}$$

then we have

$$\begin{aligned}
d(y,r)^2 &= \sum_{k=1}^{\infty} |y(k) - r(k)|^2 \\
&= \sum_{k=1}^{N} |y(k) - r(k)|^2 + \sum_{k=N+1}^{\infty} |y(k)|^2 \\
&= |y(1) - r(1)|^2 + |y(2) - r(2)|^2 + \cdots + |y(N) - r(N)|^2 + \sum_{k=N+1}^{\infty} |y(k)|^2 \\
&\leq (\frac{\epsilon}{2})^2(\frac{1}{2})^2 + (\frac{\epsilon}{4})^2(\frac{1}{2})^2 + \ldots + (\frac{\epsilon}{2^N})^2(\frac{1}{2})^2 + (\frac{\epsilon}{2})^2 \\
&\leq \frac{\epsilon^2}{4}\left(\frac{1}{4} + \frac{1}{16} + \ldots + \frac{1}{4^N}\right) + (\frac{\epsilon}{2})^2 \\
&< \frac{\epsilon^2}{4} + \frac{\epsilon^2}{4} < \frac{\epsilon^2}{2},
\end{aligned} \tag{4.17}$$

and $d(y,r) < \sqrt{\frac{\epsilon^2}{2}} < \epsilon$. Therefore, by definition 2.5.3, $\ell_2(\mathbb{Z}^+)$ is separable because there are only countably many rational sequences $r = (r(1), \ldots, r(N), 0, 0, \ldots)$ in S. □

Theorem 4.3.3. *$\ell_2(\mathbb{Z}^+)$ is a Hilbert space.*

Proof. By lemma 4.2.2, lemma 4.3.1 and lemma 4.3.2, the proof is complete. □

4.4 $L^2(\mathbb{R})$ is a Hilbert space

Lemma 4.4.1. *(Royden [7, p. 124].) Every absolutely summable series in $L^2(\mathbb{R})$ converges (in the metric space $L^2(\mathbb{R})$).*

Proof. Let $\sum_{k=1}^{\infty} f_n$ be an absolutely summable series of complex valued functions in $L^2(\mathbb{R})$, which means there exists an $M \in \mathbb{R}$ such that

$$\sum_{n=1}^{\infty} \|f_n\|_2 = M < \infty. \tag{4.18}$$

Let $g_n(x) = \sum_{k=1}^{n} |f_k(x)|$ be the sequence of partial sums of $\sum_{n=1}^{\infty} |f_n(x)|$, so

$$g_n(x) \geq 0.$$

By the Minkowski inequality (2.3.4) or the triangle inequality (Corollary 3.3.4), we have

$$\|g_n\|_2 \leq \sum_{k=1}^{n} \|f_k\|_2 \leq M. \tag{4.19}$$

So

$$\int g_n^2(x) \leq M^2.$$

However, the $g_n(x)$ are nonnegative and increasing functions of n, so

$$g(x) = \lim_{n\to\infty} g_n(x) \tag{4.20}$$

is either a nonnegative real number or ∞. Hence, by Fatou's lemma (Theorem 2.2.10),

$$\int g^2(x) \leq \liminf \int g_n^2(x) \leq M^2,$$

which means $g(x) \in L^2(\mathbb{R})$. So $g(x) < \infty$ a.e., and for almost all (fixed) $x \in \mathbb{R}$, the absolutely convergent series $\sum_{k=1}^{\infty} f_n(x)$ converges to a complex number $s(x)$. Note that $|s(x)| \leq g(x)$, and $\sum_{k=1}^{n} |f_k(x)| \leq g(x)$. Hence, we have

$$|f_k(x) - s(x)| \leq |f_k(x)| + |s(x)| \leq 2g(x).$$

By the Lebesgue Convergence Theorem (2.2.11,) $s \in L^2(\mathbb{R})$ and

$$\|f_n - s\|_2 \to 0. \tag{4.21}$$

Therefore, every absolutely summable series $\sum_{k=1}^{\infty} f_n$ in $L^2(\mathbb{R})$ converges to some $s \in L^2(\mathbb{R})$. $\square$

Theorem 4.4.2. *$L^2(\mathbb{R})$is a complete space.*

Proof. By lemmas 4.2.2 and 4.4.1, $L^2(\mathbb{R})$ is complete. $\square$

Definition 4.4.3. Let $f : \mathbb{R} \to \mathbb{C}$. Then the support of f is defined to be

$$supp(f) = \{x \in \mathbb{R} \mid f(x) \neq 0\}. \tag{4.22}$$

We say that f has "compact support" if supp$(f) \subseteq [a, b]$ for some $a, b \in \mathbb{R}$.

Lemma 4.4.4. *Suppose f is a real valued function in $L^2(\mathbb{R})$. For any $\epsilon > 0$, there exists a bounded function f_M with compact support such that $\|f - f_M\|_2 < \epsilon$.*

Proof. It is enough to show that there exists a sequence of bounded functions $\{f_n\}_{n=1}^{\infty}$ with compact support such that $\|f_n - f\|_2 \to 0$. For all $n \in \mathbb{N}$, let

$$f_n(x) = \begin{cases} 0 & \text{if } x < -n \\ -n & \text{if } -n \le x \le n, \; f(x) < -n \\ f(x) & \text{if } -n \le x \le n, \; -n \le f(x) \le n \\ n & \text{if } -n \le x \le n, \; f(x) > n \\ 0 & \text{if } x > n. \end{cases}$$

First we show that for all $x \in \mathbb{R}$, $\lim_{n\to\infty} f_n(x) = f(x)$. Fix $x \in \mathbb{R}$. Then either, $|f(x)| < \infty$ or $f(x) = \pm\infty$.

If $|f(x)| < \infty$, choose $N \ge \max\{|x|, |f(x)|\}$, and let $n \in \mathbb{N}, n > N$. Then $|x| \le N < n$ and $|f(x)| \le N < n$. So $|f_n(x) - f(x)| = |f(x) - f(x)| = 0$. Thus, for any $\epsilon > 0$, if $n > N$, then $|f_n(x) - f(x)| = 0 < \epsilon$.

On the other hand, suppose $f(x) = \pm\infty$. We take $f(x) = \infty$. ($f(x) = -\infty$ is similar). Since we want to show for every $k > 0$ there exists $N \in \mathbb{N}$ such that if $n > N$, then $f_n(x) > k$, we choose $N = \max\{|x|, k\}$, and let $n \in \mathbb{N}$ be larger than N. Since $|x| \le N$, then $|f_n(x)| = n > N \ge k$.

Next we want to show that for all $x \in \mathbb{R}$, $n \in \mathbb{N}$, $|f_n(x)| \le |f(x)|$. We have the following cases: First, if $|x| > n$, then $|f_n(x)| = 0 \le |f(x)|$. Second, if $|f(x)| > n$ and $|x| \le n$, then $|f_n(x)| = n < |f(x)|$. Third, if $|f(x)| \le n$ and $|x| \le n$, then $|f_n(x)| = |f(x)|$.

Finally, we see that $|f_n(x) - f(x)| \to 0$ and $|f_n(x) - f(x)| \le |f_n(x)| + |f(x)| \le 2|f(x)|$ where $f(x) \in L^2(\mathbb{R})$. Thus, $|f_n(x) - f(x)|^2 \le 4|f(x)|^2$ where the right hand side is summable, since $f \in L^2(\mathbb{R})$. So by the Lebesgue Convergence Theorem (2.2.11), $\|f_n(x) - f(x)\|_2 \to 0$. □

Lemma 4.4.5. *For all $a, b \in \mathbb{R}$, let $b > a$, and let $f_M : [a,b] \to \mathbb{R}$ be a bounded real-valued function. Then for all $\epsilon > 0$, there exists a simple function $\phi(x)$ such that $\|f_M - \phi\|_2 < \epsilon$.*

Proof. Choose $M \in \mathbb{N}$ such that $|f_M(x)| < M$, and choose $\{y_i\}_{i=1}^{n}$ such that $-M = y_0 < y_1 < \ldots < y_n = M$ is a partition with $|y_i - y_{i-1}| < \dfrac{\epsilon}{\sqrt{b-a}}$. Then

$$E_i = \{x \in X \mid y_{i-1} \le f_M(x) < y_i\} \cap [a,b] \tag{4.23}$$

is a measurable set in $\mathbb{R}$. Let

$$\chi_{E_i}(x) = \begin{cases} 1 & \text{if } y_{i-1} \le f_M(x) < y_i \\ 0 & \text{otherwise.} \end{cases}$$

Then let $\phi = \sum_{i=1}^{N} y_i \chi_{E_i}$. For $x \in [a,b]$ such that $y_{i-1} \le f_M(x) < y_i$, we have

$$|f_M(x) - \phi(x)| = |f_M(x) - y_i \chi_{E_i}(x)| = |f_M(x) - y_i| \le |y_{i-1} - y_i| < \frac{\epsilon}{\sqrt{b-a}}.$$

Thus,

$$\int_{\mathbb{R}} |f_M(x) - \phi(x)|^2 = \int_a^b |f_M(x) - \phi(x)|^2 < \int_a^b \frac{\epsilon^2}{b-a} = \epsilon^2.$$

□

Definition 4.4.6. Royden [7, p. 51]. Let x be a real valued function on a subset $X = [a, b]$, of $\mathbb{R}$. If there is a partition $a = x_0 < x_1 < x_2 < \ldots < x_n = b$ such that for every $i = 1, 2, \ldots, n$, ψ has only one value in the interval (x_{i-1}, x_i), then ψ is called a step function on X. Note that there exist countably many rational step functions $\psi_{\mathbb{Q}}$ such that $\psi_{\mathbb{Q}}(x) \in \mathbb{Q}$ and all $x_i \in \mathbb{Q}$.

Lemma 4.4.7. *Let E be a measurable set in $\mathbb{R}$ such that $\mu(E) < \infty$. Then for every simple function $\phi \in L^2(E)$ and $\epsilon > 0$, there exists a step function ψ such that $\|\phi - \psi\|_2 < \epsilon$.*

Proof. Since a finite linear combination of step functions is a step function, it is sufficient to consider $\phi = \chi_E$. Furthermore, by lemma 2.1.11, there exists a union of open intervals $U = (a_1, b_1) \cup (a_2, b_2) \cup \ldots \cup (a_n, b_n)$ such that if

$$S = \{x \mid x \in E \cap x \notin U\} \cup \{x \mid x \notin E \cap x \in U\}, \tag{4.24}$$

then $\mu(S) < \epsilon$.

Thus, by definition 4.4.6, since $\chi_U(x)$ can only change at

$$x \in \{a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n\},$$

$\chi_U(x)$ is a step function. However, $\chi_U(x) = 0$ for $|x| > \max\{|a_1|, \ldots, |a_n|, |b_1|, \ldots, |b_n|\}$. Therefore, we have

$$|\chi_E(x) - \chi_U(x)| = |\chi_S(x)|$$

because of the following: If $x \in E, x \in U$, then $|\chi_E(x) - \chi_U(x)| = |1 - 1| = 0$. If $x \in E, x \notin U$, then $|\chi_E(x) - \chi_U(x)| = |1 - 0| = 1$. If $x \notin E, x \in U$, then $|\chi_E(x) - \chi_U(x)| = |0 - 1| = 1$. If $x \notin E, x \notin U$, then $|\chi_E(x) - \chi_U(x)| = |0 - 0| = 0$. Then by definition 2.2.1,

$$[d(\chi_E, \chi_U)]^2 = \int |\chi_E(x) - \chi_U(x)|^2 = \int |\chi_S(x)|^2 = (\mu(S))^2 < \epsilon^2.$$

□

Lemma 4.4.8. *For every step function ψ with compact support and every $\epsilon > 0$, there exists a rational step function $\psi_{\mathbb{Q}}$ that satisfies $\|\psi - \psi_{\mathbb{Q}}\|_2 < \epsilon$.*

Proof. Ignoring a finite set $\{x_0, \ldots, x_k\}$, we can assume that

$$\psi(x) = \begin{cases} 0 & \text{if } x < x_0 \\ y_i & \text{if } x_{i-1} \leq x < x_i \ , \ 1 \leq i \leq k \\ 0 & \text{if } x \geq x_k. \end{cases}$$

Let $\epsilon > 0$. Choose $\eta = \dfrac{\sqrt{\epsilon}}{2\sqrt{x_k - x_0}} > 0$ and $\delta = \min\left\{\dfrac{1}{2}(x_i - x_{i-1}), \dfrac{\epsilon}{8(\max|y_i| + \eta)^2(k+1)}\right\}$. Moreover, choose rational numbers x_i', y_i' such that $x_i - \delta \leq x_i' \leq x_i$ and $y_i \leq y_i' \leq y_i + \eta$.

Let $M = \max(|y_i|, |y_i'|) \leq \max(|y_i| + \eta)$, and note $\delta \leq \dfrac{\epsilon}{8M^2(k+1)}$. Let

$$\psi_{\mathbb{Q}}(x) = \begin{cases} 0 & \text{if } x < x_0' \\ y_i' & \text{if } x_{i-1}' \leq x < x_i' \text{ , } 1 \leq i \leq k \\ 0 & \text{if } x \geq x_k'. \end{cases}$$

Because $0 < \delta \leq \min\left\{\dfrac{1}{2}(x_i - x_{i-1})\right\}$, $x_0' \leq x_0 \leq x_1' \leq x_1 \leq \ldots \leq x_{k-1} \leq x_k' \leq x_k$. Then, for $x < x_0'$ or $x \geq x_k$, we have

$$|\psi(x) - \psi_{\mathbb{Q}}(x)| = |0 - 0| = 0. \tag{4.25}$$

Next, for $x_i' \leq x < x_i$ (a segment of width at most δ),

$$|\psi(x) - \psi_{\mathbb{Q}}(x)| \leq |\psi(x)| + |\psi_{\mathbb{Q}}(x)| \leq 2M. \tag{4.26}$$

Finally, for $x_{i-1}' \leq x_{i-1} < x < x_i' \leq x_i$,

$$|\psi(x) - \psi_{\mathbb{Q}}(x)| = |y_i - y_i'| \leq \eta. \tag{4.27}$$

Then

$$\begin{aligned}
\|\psi(x) - \psi_{\mathbb{Q}}(x)\|_2^2 &= \int_{\mathbb{R}} |\psi(x) - \psi_{\mathbb{Q}}(x)|^2 \\
&= \sum_{i=0}^{k} \int_{x_i'}^{x_i} |\psi(x) - \psi_{\mathbb{Q}}(x)|^2 + \sum_{i=1}^{k} \int_{x_{i-1}}^{x_i'} |\psi(x) - \psi_{\mathbb{Q}}(x)|^2 \\
&\leq \sum_{i=0}^{k} \int_{x_i'}^{x_i} 4M^2 + \sum_{i=1}^{k} \int_{x_{i-1}}^{x_i'} \eta^2 \\
&= \sum_{i=0}^{k} \int_{x_i'}^{x_i} 4M^2 + \sum_{i=1}^{k} \int_{x_{i-1}}^{x_i'} \frac{\epsilon}{4(x_k - x_0)} \\
&\leq \sum_{i=0}^{k} 4\delta M^2 + \int_{x_0}^{x_k'} \frac{\epsilon}{4(x_k - x_0)} \\
&\leq \sum_{i=0}^{k} \frac{\epsilon}{2(k+1)} + \int_{x_0}^{x_k} \frac{\epsilon}{4(x_k - x_0)} \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon(x - x_0)}{4(x - x_0)} < \epsilon.
\end{aligned} \tag{4.28}$$

□

Theorem 4.4.9. *For every real valued $f \in L^2(\mathbb{R})$ there exists a rational step function $\psi_{\mathbb{Q}}$ such that $\|f - \psi_{\mathbb{Q}}\|_2 < \epsilon$. Furthermore, there exist countably many rational step functions $\psi_{\mathbb{Q}}$.*

Proof. By lemmas 4.4.4, 4.4.5, 4.4.7 and 4.4.8 respectively, given $f \in L^2(\mathbb{R})$ and $\epsilon > 0$ there exist a bounded function f_M, a simple function ϕ, a step function ψ, and a rational step function $\psi_{\mathbb{Q}}$ in $L^2(\mathbb{R})$ such that

$$\|f - f_M\|_2, \|f_M - \phi\|_2, \|\phi - \psi\|_2, \text{ and } \|\psi - \psi_{\mathbb{Q}}\|_2$$

are all less than $\frac{\epsilon}{5}$. Then by the triangle inequality (Corollary 3.3.4) we have

$$\begin{aligned} \|f - \psi_{\mathbb{Q}}\|_2 &\leq \|f - f_M\|_2 + \|f_M - \phi\|_2 + \|\phi - \psi\|_2 + \|\psi - \psi_{\mathbb{Q}}\|_2 \\ &< \frac{\epsilon}{5} + \frac{\epsilon}{5} + \frac{\epsilon}{5} + \frac{\epsilon}{5} < \epsilon. \end{aligned} \tag{4.29}$$

□

Theorem 4.4.10. *For every complex-valued function $f \in L^2(\mathbb{R})$, there exist countably many complex-valued rational step functions $\psi_{\mathbb{Q}}$ such that $\|f - \psi_{\mathbb{Q}}\|_2 < \epsilon$.*

Proof. Let $f = u + iv$ be a complex valued function in $L^2(\mathbb{R})$. By theorem 4.4.9, for $\epsilon > 0$, there exist real-valued rational step functions $\alpha, \gamma \in L^2(\mathbb{R})$ such that $d(u, \alpha) < \frac{\epsilon}{2}$ and $d(v, \gamma) < \frac{\epsilon}{2}$. Now if $\psi_{\mathbb{Q}} = \alpha + i\gamma$, then we have

$$\begin{aligned} [d(f, \psi_{\mathbb{Q}})]^2 &= \|f - \psi_{\mathbb{Q}}\|_2^2 \\ &= \int_{\mathbb{R}} |f - \psi_{\mathbb{Q}}|^2 \\ &= \int_{\mathbb{R}} |u - \alpha + i(v - \gamma)|^2 \\ &= \int_{\mathbb{R}} |u - \alpha|^2 + \int_{\mathbb{R}} |v - \gamma|^2 \\ &= \left(\frac{\epsilon}{2}\right)^2 + \left(\frac{\epsilon}{2}\right)^2 \\ &= \frac{\epsilon^2}{4} + \frac{\epsilon^2}{4} = \frac{\epsilon^2}{2} < \epsilon^2. \end{aligned} \tag{4.30}$$

So $d(f, \psi) < \epsilon$. □

Theorem 4.4.11. *$L^2(\mathbb{R})$ is a Hilbert space.*

Proof. By theorem 4.4.2, $L^2(\mathbb{R})$ is complete (Definition 2.5.7). Furthermore, by theorem 4.4.9, and 4.4.10, $L^2(\mathbb{R})$ is separable (Definition 2.5.3). Therefore, by definition 4.1.1, $L^2(\mathbb{R})$ is a Hilbert space. □

4.5 Hilbert spaces have orthonormal bases

Definition 4.5.1. (Messer [5, p. 104].) Let $S = \{v_i\}_{i=1}^n$ be a finite subset of a vector space V. We say that S is linearly independent if $\sum_{i=1}^n a_i v_i = 0$ implies that $a_i = 0$ for each a_i.

Definition 4.5.2. Let $S = \{v_i\}_{i=1}^\infty$ be an infinite subset of a vector space V. We say that S is linearly independent if any finite subset of S is linearly independent.

Definition 4.5.3. (Dym and McKean [2, p. 23].) Let $S = \{v_i\}_{i=1}^\infty$ be an infinite subset of an inner product space V. We define span S by saying that $v \in$ span S if and only if there exist $\{x_i\}_{i=1}^\infty \subseteq S$ and $c_i(n) \in \mathbb{C}$ (depending on both i and n) such that

$$\lim_{n\to\infty} \left\| v - \sum_{i=1}^{n} c_i(n) x_i \right\| = 0. \tag{4.31}$$

If span $S = V$, we say S spans V.

Definition 4.5.4. (Dym and McKean [2, p. 23].) If $S = \{v_i\}_{i=1}^\infty \subseteq V$, an inner product space, is linearly independent and spans V, then S is a basis for V.

Lemma 4.5.5. *(Dym and McKean [2, p. 24].) For any countable linearly independent set $S \subseteq V$, an inner product space, there exists an orthonormal set $B = \{e_i\}_{i=1}^\infty \subseteq V$ such that span B =span S.*

Proof. If $S = \{g_1, g_2, \ldots\}$ is a linearly independent set of functions, then by the Gram-Schmidt process we have an orthonormal set

$$\begin{aligned} e_1 &= \frac{g_1}{\|g_1\|}, \\ e_2 &= \frac{g_2 - \langle g_2, e_1\rangle e_1}{\|g_2 - \langle g_2, e_1\rangle e_1\|}, \\ &\vdots \\ e_n &= \frac{g_n - \sum_{k<n}\langle g_n, e_k\rangle e_k}{\left\|g_n - \sum_{k<n}\langle g_n, e_k\rangle e_k\right\|}, \\ &\vdots \end{aligned}$$

At each step we have span B = span S. If $\left\|g_n - \sum_{k<n}\langle g_n, e_k\rangle e_k\right\| = 0$, then by the properties of norm we have

$$g_n - \sum_{k<n}\langle g_n, e_k\rangle e_k = 0, \text{ or } g_n = \sum_{k<n}\langle g_n, e_k\rangle e_k, \tag{4.32}$$

which means g_n is a linear combination of $\{g_1, g_2, \ldots, g_{n-1}\}$. This is a contradiction because $\{g_1, g_2, \ldots, g_n\}$ is a linearly independent set. $\square$

Lemma 4.5.6. *Let V be an inner product space. For any countable set $C \subseteq V$, there exists a countable linearly independent set $S \subseteq C$ such that span S = span C.*

Proof. Let $C = \{v_i\}_{i=1}^{\infty} \subseteq V$ be a countable set, and let $C_n = \{v_i\}_{i=1}^{n}$. We choose $S = C$ inductively as follows: Let $S_0 = \emptyset$, which is always linearly independent. Suppose S_n is linearly independent, and span S_n = span C_n.

Note that $C_{n+1} = C_n \cup \{v_{n+1}\}$. There are two cases:

(1) $v_{n+1} \in$ span C_n = span S_n. Then span C_{n+1} = span C_n. So let $S_{n+1} = S_n$. Since S_n is linearly independent, then S_{n+1} is linearly independent. Thus, span (C_{n+1}) =span C_n = span S_n = span S_{n+1}.

(2) $v_{n+1} \notin$ span C_n = span S_n. Then let $S_{n+1} = S_n \cup \{v_{n+1}\}$. So by the Expansion Lemma (Messer [5, p. 117]), S_{n+1} is linearly independent. Since $S_{n+1} \subseteq C_{n+1}$, span $(S_{n+1}) \subseteq$ span C_{n+1}. Assume $v \in$ span C_{n+1}. Then for some $c_k \in \mathbb{C}$,

$$v = \sum_{k=1}^{n+1} c_k v_k = \sum_{k=1}^{n} c_k v_k + c_{n+1} v_{n+1}. \tag{4.33}$$

Since $\sum_{k=1}^{n} c_k v_k \in$ span $C_n \subseteq$ span S_{n+1} and $c_{n+1}v_{n+1} \in$ span S_{n+1}, then $v = \sum_{k=1}^{n} c_k v_k + c_{n+1}v_{n+1} \in$ span S_{n+1}. Therefore, span C_{n+1} = span S_{n+1}.

Let $S = \cup_{n=1}^{\infty} S_n$. We want to show:

(1) S is linearly independent.

(2) span S = span C.

(1) Let B be a finite subset of S. Since $B \subseteq S_k$ for some $k \in \mathbb{N}$, and S_k is linearly independent, then B is linearly independent. So by definition 4.5.1, S is linearly independent.

(2) Since $S \subseteq C$, span $S \subseteq$ span C.

Conversely, suppose $v \in$ span C. By definition 4.5.3, there exists $u_n \in$ span C_n, such that $\lim_{n\to\infty} \|v - u_n\| = 0$. However, $u_n \in$ span S_n = span C_n, so $\lim_{n\to\infty} \|v - u_n\| = 0$ for some $u_n \in$ span S_n. So $v \in$ span S by definition 4.5.3.

□

Theorem 4.5.7. *(Dym and McKean [2, p. 25].) Every Hilbert space H has a countable orthonormal basis.*

Proof. Let H be a Hilbert space, and let C be any countable dense (Definition 2.5.2) subset of H such that span $C = H$. For any $v \in H$, let

$$c_i(n) = \begin{cases} 1 & \text{if } i = n \\ 0 & \text{if } i \neq n, \end{cases}$$

and let $\{x_i\}_{i=1}^{\infty}$ be elements of C such that $\lim_{n\to\infty} x_n = v$. Then

$$\lim_{n\to\infty} \left\| v - \sum_{i=1}^{n} c_i(n) x_i \right\| = \lim_{n\to\infty} \|v - x_n\| = 0$$

So there exist $c_i(n) \in \mathbb{C}$ and $\{x_i\}_{i=1}^{\infty} \subseteq C$ such that

$$\lim_{n\to\infty} \left\| v - \sum_{i=1}^{n} c_i(n) x_i \right\| = 0.$$

Thus, by definition 4.5.3, $v \in$ span C. This means $H \subseteq$ span C. So $H =$ span C. By lemma 4.5.6, there exists a countable linearly independent set $S \subset C$ such that span $S =$ span C=H. Furthermore, by Lemma 4.5.5, there exists an orthonormal set $B = \{e_1, e_2, \ldots\} \subset H$ such that span $B =$ span $S = H$. Therefore, H has a countable orthonormal basis B. □

Lemma 4.5.8. *Let H be a Hilbert space, and let $B = \{e_i\}_{i=1}^{\infty}$ be an orthonormal subset of H. Then $\|v - \sum_{i=1}^{\infty} c_i(n) e_i\|$ is minimized if and only if $c_i(n) = \langle v, e_i \rangle$.*

Proof. (Dym and McKean [2, p. 23, 25].) Let $B = \{e_i\}_{i=1}^{\infty} \subseteq H$ be orthonormal. Then for every $v \in H$ and $c_i(n) \in \mathbb{C}$, we have the following expansion:

$$\begin{aligned} \left\| v - \sum_{i=1}^{n} c_i(n) e_i \right\|^2 &= \left\| v - \sum_{i=1}^{n} \langle v, e_i \rangle e_i + \sum_{i=1}^{n} \left[\langle v, e_i \rangle - c_i(n) \right] e_i \right\|^2 \\ &= \left\| v - \sum_{i=1}^{n} \langle v, e_i \rangle e_i \right\|^2 \\ &\quad + 2Re \left\langle v - \sum_{i=1}^{n} \langle v, e_i \rangle e_i, \sum_{j=1}^{n} \left[\langle v, e_j \rangle - c_j(n) \right] e_j \right\rangle \\ &\quad + \left\| \sum_{i=1}^{n} \left[\langle v, e_i \rangle - c_i(n) \right] e_i \right\|^2 . \end{aligned} \tag{4.34}$$

However, the middle expression is the real part of a sum of constant multiples of terms like

$$\begin{aligned} \left\langle v - \sum_{i=1}^{n} \langle v, e_i \rangle e_i, e_j \right\rangle &= \langle v, e_j \rangle - \left\langle \sum_{i=1}^{n} \langle v, e_i \rangle e_i, e_j \right\rangle \\ &= \langle v, e_j \rangle - \sum_{i=1}^{n} \left(\langle v, e_i \rangle \right) \left(\langle e_i, e_j \rangle \right) \\ &= \langle v, e_j \rangle - \langle v, e_j \rangle = 0. \end{aligned} \tag{4.35}$$

So we have

$$\left\| v - \sum_{i=1}^{n} c_i(n) e_i \right\|^2 = \left\| v - \sum_{i=1}^{n} \langle v, e_i \rangle e_i \right\|^2 + \left\| \sum_{i=1}^{n} \left[\langle v, e_i \rangle - c_i(n) \right] e_i \right\|^2, \tag{4.36}$$

where by lemma 3.6.2,

$$\begin{aligned} \left\| \sum_{i=1}^{n} \left[\langle v, e_i \rangle - c_i(n) \right] e_i \right\|^2 &= \left\langle \sum_{i=1}^{n} \left[\langle v, e_i \rangle - c_i(n) \right] e_i, \sum_{i=1}^{n} \left[\langle v, e_i \rangle - c_i(n) \right] e_i \right\rangle \\ &= \sum_{i=1}^{n} \left| \langle v, e_i \rangle - c_i(n) \right|^2 . \end{aligned} \tag{4.37}$$

So

$$\left\| v - \sum_{i=1}^{n} c_i(n) e_i \right\|^2 = \left\| v - \sum_{i=1}^{n} \langle v, e_i \rangle e_i \right\|^2 + \sum_{i=1}^{n} \left| \langle v, e_i \rangle - c_i(n) \right|^2. \tag{4.38}$$

Therefore,

$$\left\| v - \sum_{i=1}^{n} \langle v, e_i \rangle e_i \right\| \leq \left\| v - \sum_{i=1}^{n} c_i(n) e_i \right\|, \tag{4.39}$$

and if any $\langle v, e_i \rangle - c_i(n) \neq 0$, the inequality is strict. □

Theorem 4.5.9. *If $B = \{e_i\}_{i=1}^{\infty}$ is an orthonormal basis for a Hilbert space H, then for all $v \in H$,*

$$\lim_{n \to \infty} \left\| v - \sum_{i=1}^{n} \langle v, e_i \rangle e_i \right\| = 0. \tag{4.40}$$

Proof. Let $v \in H$. By definition 4.5.4, there exist $c_i(n) \in \mathbb{C}$ such that

$$\lim_{n \to \infty} \left\| v - \sum_{i=1}^{n} c_i(n) e_i \right\| = 0. \tag{4.41}$$

Since

$$\left\| v - \sum_{i=1}^{n} \langle v, e_i \rangle e_i \right\| \leq \left\| v - \sum_{i=1}^{n} c_i(n) e_i \right\|, \tag{4.42}$$

by the Squeeze Theorem we have

$$\lim_{n \to \infty} \left\| v - \sum_{i=1}^{n} \langle v, e_i \rangle e_i \right\| = 0. \tag{4.43}$$

□

CHAPTER 5

CLOSED SUBSPACES AND BOUNDED LINEAR FUNCTIONS

5.1 Subsets of inner product spaces

Definition 5.1.1. (Griffel [3, p. 144].) Let S be a subset of an inner product space V. If there is a number M that for all $s \in S$, $\|s\| \leq M$, then S is said to be a bounded set.

Definition 5.1.2. (Griffel [3, p. 99].) Let S be a subset of an inner product space V. We say x is a limit point of S if for some sequence $\{s_n\}_{n=1}^{\infty}$ in S, $\lim_{n\to\infty} s_n = x$.

Theorem 5.1.3. *Let V be an inner product space and $S \subseteq V$. Then the following are equivalent:* 1. *x is a limit point of S.* 2. *For every $\epsilon > 0$ there exists $s \in S$ such that $\|x - s\| < \epsilon$.*

Proof. $1 \Rightarrow 2$. For $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that if $n > N$, then $\|s_n - x\| < \epsilon$. Let $s = s_{N+1}$. Then $\|x - s\| = \|x - s_{N+1}\| < \epsilon$.

$2 \Rightarrow 1$. Since for any $\epsilon > 0$ there exists $s \in S$ such that $\|x - s\| < \epsilon$, for $n \in \mathbb{N}$ choose $s_n \in S$ such that $\|x - s_n\| < \frac{1}{n}$. Given $\epsilon > 0$, choose $N > \frac{1}{\epsilon}$. If we assume $n > N$ where $n \in \mathbb{N}$, then $\|x - s_n\| < \frac{1}{n} < \frac{1}{N} < \epsilon$. □

Definition 5.1.4. (Griffel [3, p. 100].) We say S, a subset of an inner product space V, is closed if every convergent sequence $\{s_n\}_{n=1}^{\infty}$ in S converges to $x \in S$.

Definition 5.1.5. (Griffel [3, p. 101].) Let V be an inner product space, and $S \subseteq V$. Then the closure of S is

$$\overline{S} = S \cup \{x \in V \mid x \text{ is a limit point of } S.\} \tag{5.1}$$

Theorem 5.1.6. *Let S be a subset of an inner product space V. Then S is closed if and only if $S = \overline{S}$.*

Proof. Suppose S is a closed subset of V. Then by definition 5.1.4, every sequence $\{s_n\}_{n=1}^{\infty} \in S$ converges to $x \in S$. Thus, by definition 5.1.5, $S = \overline{S}$.

Conversely, suppose $\overline{S} = S$. Since by definition 5.1.5, every convergent sequence $\{s_n\}_{n=1}^{\infty} \subseteq S$ converges to a limit point $x \in \overline{S} = S$, then S is closed by definition 5.1.4. □

Definition 5.1.7. (Griffel [3, p. 149].) Let V be an inner product space, and $S \subseteq V$. We say S is compact if every infinite sequence $\{s_n\}_{n=1}^{\infty}$ of elements of S has a subsequence $\{s_{n_k}\}_{k=1}^{\infty}$ that converges to some $x \in S$.

Definition 5.1.8. (Griffel [3, p. 152].) We say that S, a subset of an inner product space V, is relatively compact if $\overline{S}$ is compact.

Theorem 5.1.9. *(Griffel [3, p. 152].) Let V be an inner product space, and $S \subseteq V$. The following are equivalent. 1. S is relatively compact. 2. Every sequence $\{s_n\}_{n=1}^{\infty} \subseteq S$ has a subsequence $\{s_{n_k}\}_{k=1}^{\infty}$ that converges to $x \in \overline{S}$.*

Proof. $1 \Rightarrow 2$. Let $S \subseteq V$ be relatively compact, so by definition 5.1.8, $\overline{S}$ is compact. Suppose we have a sequence $\{s_n\}_{n=1}^{\infty} \subseteq S$, which is also a sequence in $\overline{S}$, so by definition 5.1.7 there is a subsequence $\{s_{n_k}\}_{k=1}^{\infty}$ that converges to $x \in \overline{S}$.

$2 \Rightarrow 1$. We want to show that $\overline{S}$ is compact, so we assume $\{\overline{s_n}\}_{n=1}^{\infty} \subseteq \overline{S}$. By theorem 5.1.3, choose $\{s_n\}_{n=1}^{\infty} \subseteq S$ such that

$$\|s_n - \overline{s_n}\| < \frac{1}{n}. \tag{5.2}$$

By (2) there exists a subsequence $\{s_{n_k}\}_{k=1}^{\infty}$ that converges to a point x in $\overline{S}$. We want to show that

$$\lim_{n \to \infty} \overline{s_{n_k}} = x. \tag{5.3}$$

Suppose $\epsilon > 0$. Since $\lim_{k \to \infty} s_{n_k} = x$, we can choose N_2 such that if $k > N_2$ then $\|s_{n_k} - x\| < \frac{\epsilon}{2}$, and we also choose $N = \max\{N_1, N_2\}$ where $N_1 = \frac{2}{\epsilon}$. Let $k \in \mathbb{N}$, $k > N$. Then

$$\begin{aligned} \|\overline{s_{n_k}} - x\| &\leq \|\overline{s_{n_k}} - s_{n_k}\| + \|s_{n_k} - x\| \\ &< \frac{1}{n_k} + \frac{\epsilon}{2} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned} \tag{5.4}$$

Therefore, $\overline{S}$ is compact. □

Theorem 5.1.10. *(Griffel [3, p. 150].) In an inner product space V every compact set S is closed and bounded.*

Proof. Suppose $S \subseteq V$ is compact. By definition 5.1.7, every sequence $\{s_n\}_{n=1}^{\infty} \subseteq S$ has a subsequence $\{s_{n_k}\}_{k=1}^{\infty}$ that converges to a limit point $s \in S$, so any convergent sequence must converge to a point in S. Thus, by definition 5.1.4, S is closed. Also, S must be bounded; otherwise, for every $N \in \mathbb{N}$, the norm of some element of S is larger than N. This means for all $N \in \mathbb{Z}^+$, there is $s_n \in S$ such that $\|s_n\| > N$, so no subsequence of $\{s_n\}_{n=1}^{\infty}$ converges. □

Theorem 5.1.11. *Heine-Borel theorem:*

(Griffel [3, p. 150, 154].) In a finite-dimensional inner product space V a subset S is compact if and only if S is closed and bounded.

Proof. See Griffel [3, p. 150,154], and compare with Theorem 5.1.10 and Theorem 5.3.4. □

5.2 Linear functions on Hilbert spaces

Definition 5.2.1. (Messer [5, p. 213].) Let H_1, H_2 be vector spaces. We say that $A : H_1 \to H_2$ is a linear function if for all $u, v \in H_1$ and all $c \in \mathbb{C}$

$$A(u+v) = A(u) + A(v), \ \ A(cu) = cA(u). \tag{5.5}$$

Whenever $H_1 = H_2$, we say A is a linear operator.

Definition 5.2.2. (Griffel [3, p. 196]). Let H_1, H_2 be inner product spaces and $A : H_1 \to H_2$ be a function. If for every $\epsilon > 0$, there exists $\delta > 0$ such that $\|A(x+h) - A(x)\| < \epsilon$ whenever $\|h\| < \delta$, then we say that A is continuous at x. Moreover, if A is continuous at all $x \in H$, we say A is continuous.

Definition 5.2.3. (Griffel [3, p. 211]). Let H_1, H_2 be inner product spaces and let $A : H_1 \to H_2$ be linear. If there is a number M such that $\|A(x)\| \leq M\|x\|$ for all $x \in H$, then A is a bounded linear function.

Theorem 5.2.4. *Let H_1, H_2 be inner product spaces, and let $A : H_1 \to H_2$ be a linear function. The following are equivalent:* (1) *A is bounded.* (2) *A is continuous.*

Proof. 1 $\Rightarrow$ 2. Suppose there exists an $M \in \mathbb{R}$ such that $\|A(v)\| \leq M\|v\|$ for all $v \in H_1$. Assume $u \in H_1$ and $\epsilon > 0$. Choose $\delta = \frac{\epsilon}{M}$. If $v \in H_1$ and $\|u - v\| < \delta$, then $\|Au - Av\| = \|A(u-v)\| \leq M\|u-v\| < M\delta = M(\frac{\epsilon}{M}) = \epsilon$.

2 $\Rightarrow$ 1. Suppose for all $u \in H_1$ and $\epsilon > 0$, there exists $\delta > 0$ such that if $v \in H_1$, $\|u - v\| < \delta$, then $\|A(u) - A(v)\| < \epsilon$. Let $u = 0$ and $\epsilon = 1$. Since A is continuous at 0, there exists $\delta > 0$ such that if $v \in H_1$, $\|v\| < \delta$, then $\|Av\| < 1$. Choose $M = \frac{2}{\delta}$. Then, for all $v \in H_1$,

$$\left\| \frac{v}{M\|v\|} \right\| = \frac{\|v\|}{M\|v\|} = \frac{1}{M} < \delta, \tag{5.6}$$

so

$$\frac{\|A(v)\|}{M\|v\|} = \left\| A\left(\frac{v}{M\|v\|}\right) \right\| < 1. \tag{5.7}$$

So there exists an $M \in \mathbb{R}$ such that $\|A(v)\| \leq M\|v\|$ for all $v \in H_1$. □

Definition 5.2.5. (Griffel [3, p. 212]). Let H_1, H_2 be inner product spaces, and let $A : H_1 \to H_2$ be a bounded linear function. We define

$$\|A\| = \sup\left\{\frac{\|Av\|}{\|v\|} : v \in H_1, v \neq 0\right\}. \tag{5.8}$$

Consequently, for all $v \in H_1$,

$$\|Av\| \leq \|A\|\|v\|. \tag{5.9}$$

Note that A is bounded if and only if $\|A\|$ is finite and $\|A\| = 0$ if and only if $Av = 0$ for all $v \in H$.

Theorem 5.2.6. *(Griffel [3, p. 212]). Let H_1, H_2 be inner product spaces, and let $A : H_1 \to H_2$ be a bounded linear function. Then*

$$\|A\| = \sup_{\|v\|=1} \|Av\|. \tag{5.10}$$

Proof. By definition 5.2.5, if $\|v\| = 1$, then $\|Av\| \leq \|A\|\|v\|$, which means

$$\|Av\| \leq \|A\|. \tag{5.11}$$

Conversely, suppose $\epsilon > 0$. We show that $\|A\| - \epsilon$ is not an upper bound for

$$\{\|Av\| \mid \|v\| = 1\}.$$

So we want v such that $\|v\| = 1$, $\|Av\| > \|A\| - \epsilon$. By definition 5.2.5, choose a nonzero $w \in H$ such that $\dfrac{\|Aw\|}{\|w\|} > \|A\| - \epsilon$. Let $v = \dfrac{w}{\|w\|}$. Then $\|v\| = 1$ and $\|Av\| = \left\|A\dfrac{w}{\|w\|}\right\|$. Since A is linear, $\|Av\| = \dfrac{\|Aw\|}{\|w\|} > \|A\| - \epsilon$. □

5.3 Continuity of norms and inner products

Definition 5.3.1. For any $v \in H$, $L_v : H \to \mathbb{C}$ is given by $L_v(x) = \langle x, v\rangle$.

Theorem 5.3.2. *Let $L_v(x) = \langle x, v\rangle$. Then L_v is a bounded linear function (and so L_v is continuous).*

Proof. $L_v(x)$ is linear because for all $u, v, x \in H$ and $\alpha \in \mathbb{C}$ we have

$$L_v(x + y) = \langle x + y, v\rangle = \langle x, v\rangle + \langle y, v\rangle = L_v(x) + L_v(y). \tag{5.12}$$

$$\alpha L_v(x) = \alpha\langle x, v\rangle = \langle \alpha x, v\rangle = L_v(\alpha x). \tag{5.13}$$

By the Schwarz inequality,

$$|\langle x, v\rangle| \leq \|x\|\|v\|, \tag{5.14}$$

so $\|L_v(x)\| \leq \|v\|$. Thus, $L_v(x)$ is a bounded linear function. □

Corollary 5.3.3. *Suppose v, x belong to an inner product space V. Then $\|.\| : V \longrightarrow \mathbb{R}$ is continuous.*

Proof. We want to show that $\lim_{\|x\|\to 0} \|v + x\| = \|v\|$ for any $x, v \in V$. So we have

$$\|v + x\|^2 = \langle v + x, v + x\rangle = \|v\|^2 + 2\text{Re } \langle x, v\rangle + \|x\|^2. \tag{5.15}$$

Since Re $(L_v(x))$ is continuous (Theorem 5.3.2) at x, $\|v + x\| \longrightarrow \|v\|$ as $\|x\| \longrightarrow 0$. Thus, $\|.\| : V \longrightarrow \mathbb{R}$ is continuous. □

Theorem 5.3.4. *Let $S = \{v \in \ell_2(\mathbb{Z}^+) \mid \|v\| \leq 1\} \subseteq \ell_2(\mathbb{Z}^+)$. Then S is closed and bounded, but not compact.*

Compare Theorems 5.1.10, and 5.1.11.

Proof. In $\ell_2(\mathbb{Z}^+)$, consider the unit ball

$$S = \{v \in \ell_2(\mathbb{Z}^+) \mid \|v\| \leq 1\} \subseteq \ell_2(\mathbb{Z}^+). \tag{5.16}$$

Clearly, S is bounded by 1. It must also be closed; to see this, let $\{s_n\}_{n=1}^{\infty}$ be a sequence in S such that as $n \longrightarrow \infty$, $s_n \longrightarrow s$. We see that by continuity of $\|.\|$, $\|s\| = \lim_{n\to\infty} \|s_n\| \leq 1$, so $\|s\| \in S$.

However, the sequence $\{e_i\}_{i=1}^{\infty}$ with $e_i = (0, 0, \ldots, 1, 0, \ldots)$ is a sequence in S. Note that for all $i \neq j$,

$$\|e_i - e_j\| = \sqrt{(1)^2 + (1)^2} = \sqrt{2}, \tag{5.17}$$

which means neither this sequence nor any of its subsequences can be Cauchy. Therefore, $\{e_i\}_{i=1}^{\infty}$ has no convergence subsequence, so S is not compact. □

5.4 Riesz-Fischer theorem in finite and infinite dimension spaces

Lemma 5.4.1. *(Griffel [3, p. 261, 262]). Let V be an inner product space, $\{e_k\}_{k=1}^{N}$ be an orthonormal subset of v, and let sequence $\{c_k\}_{k=1}^{N} \subseteq \mathbb{C}$. Then $\left\|\sum_{k=1}^{N} c_k e_k\right\|^2 = \sum_{k=1}^{N} |c_k|^2$.*

Proof. This follows from Lemma 3.6.2. □

Lemma 5.4.2. *(Griffel [3, p. 191]). Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal subset of a Hilbert space H, and let $\{c_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{C}$. Then the following are equivalent:*

(1) $\sum_{n=1}^{\infty} c_n e_n$ converges to an element of H.

(2) $\{c_n\}_{n=1}^{\infty} \in \ell_2(\mathbb{Z}^+)$.

Furthermore,

$$\left\|\sum_{n=1}^{\infty} c_n e_n\right\|^2 = \sum_{n=1}^{\infty} |c_n|^2 \tag{5.18}$$

if (1), (2) *are both true.*

Proof. By the Cauchy Criterion, $\sum_{n=1}^{\infty} c_n e_n$ converges if and only if $\left\|\sum_{k=n}^{N} c_k e_k\right\|^2 \to 0$. Furthermore, $\sum_{n=1}^{\infty} |c_n|^2$ converges if and only if $\sum_{k=n}^{N} |c_k|^2 \to 0$ by Cauchy. However, by Lemma 5.4.1,

$$\left\|\sum_{k=n}^{N} c_k e_k\right\|^2 = \sum_{k=n}^{N} |c_k|^2 . \tag{5.19}$$

So, 1 $\iff$ 2.

Suppose 1 and 2 hold. Then by the continuity of norm,

$$\begin{aligned}\left\|\sum_{n=1}^{\infty} c_n e_n\right\|^2 &= \lim_{N\to\infty} \left\|\sum_{n=1}^{N} c_n e_n\right\|^2 \\ &= \lim_{N\to\infty} \sum_{n=1}^{N} |c_n|^2 \text{ by lemma 5.4.1,} \\ &= \sum_{n=1}^{\infty} |c_n|^2 .\end{aligned} \tag{5.20}$$

□

Theorem 5.4.3. *The Risez-Fischer theorem : (Griffel [3, p. 191]). Let* $\{e_k\}_{k=1}^{n}$ *be an orthonormal subset of a Hilbert space* H*. Then span* $\{e_k\}_{k=1}^{n} = S$ *is isomorphic to the Hilbert space* $\mathbb{C}^n$*.*

Proof. We define $T : \mathbb{C}^n \to S$ by

$$T(c_1, c_2, \ldots, c_n) = \sum_{k=1}^{n} c_k e_k. \tag{5.21}$$

T is clearly linear and bijective, and by Lemma 5.4.1, T is norm-preserving. So T is an isomorphism by Lemma 3.2.2. □

Theorem 5.4.4. *(Griffel [3, p. 191]). Let* $\{e_n\}_{n=1}^{\infty}$ *be an orthonormal subset of an Hilbert space* H*. Then span* $\{e_n\}_{n=1}^{\infty} = S$ *is isomorphic to the Hilbert space* $\ell_2(\mathbb{Z}^+)$*.*

Proof. Assume span $\{e_i\}_{i=1}^{n} \subseteq H$. Then

$$T(\{c_i\}_{i=1}^{\infty}) = \sum_{k=1}^{\infty} c_k e_k \tag{5.22}$$

is well defined and onto by Lemma 5.4.2. Clearly, T is linear, and $T^{-1}(c_1, c_2, \ldots) = \sum_{k=1}^{\infty} c_n e_n$. Finally, by Lemma 5.4.2, T is norm preserving. So T is an isomorphism by Lemma 3.2.2. □

Corollary 5.4.5. *Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal subset of an Hilbert space H. Span $\{e_n\}_{n=1}^{\infty} = S$ is a subspace of H.*

Proof. In the notation of theorem 5.4.4, we see that $T^{-1}(\ell_2(\mathbb{Z}^+)) = S$. Since the image of a vector space under a linear map is a subspace, then S is a subspace in H. □

Corollary 5.4.6. *Any finite dimension Hilbert space H_1 is isomorphic to $\mathbb{C}^n$, and any infinite dimension Hilbert space H_2 is isomorphic to $\ell_2(\mathbb{Z}^+)$.*

Proof. Any Hilbert space has an orthonormal basis. There are two cases:

(1) If the orthonormal basis $B_1 = \{e_k\}_{k=1}^{n}$ is a finite set, then by theorem 5.4.3, span B_1 is isomorphic to Hilbert space $\mathbb{C}^n$ with a finite dimension.

(2) If the orthonormal basis $B_2 = \{e_k\}_{k=1}^{\infty}$ is a infinite set, then by theorem 5.4.4, span B_2 is isomorphic to the Hilbert space $\ell_2(\mathbb{Z}^+)$ with infinite dimension.

□

5.5 Closed subspaces of Hilbert spaces are Hilbert spaces

Definition 5.5.1. (Dym and McKean [2, p. 28].) Let W be a subspace of a Hilbert space H. We say that

$$W^{\perp} = \left\{ w^{\perp} \in H \mid \langle w^{\perp}, w \rangle = 0 \text{ for any } w \in W \right\}$$

is the orthogonal complement of W.

Lemma 5.5.2. *(Dym and McKean [2, p. 29].) If W is a subspace of a Hilbert space H, then $W^{\perp}$ is a closed subspace of H.*

Proof. For every $u^{\perp}, w^{\perp} \in W^{\perp}, w \in W$, and any complex number α, we have

$$\langle u^{\perp} + w^{\perp}, w \rangle = \langle u^{\perp}, w \rangle + \langle w^{\perp}, w \rangle = 0, \tag{5.23}$$

and

$$\langle \alpha u^{\perp}, w \rangle = \alpha \langle u^{\perp}, w \rangle = 0 \tag{5.24}$$

since any $w \in W$ is orthogonal to any element in $W^{\perp}$. So $W^{\perp}$ is a subspace of H.

In addition, let $\{w_n^\perp\}_{n=1}^\infty$ be a sequence in $W^\perp$ such that $\lim_{n\to\infty} w_n^\perp = x \in H$. We want to show that $x \in W^\perp$. For any $w \in W$, we have a continuous linear function $L_w(x) = \langle x, w\rangle$, so

$$\begin{aligned}\langle x, w\rangle &= L_w\left(\lim_{n\to\infty}(w_n^\perp)\right)\\ &= \lim_{n\to\infty} L_w(w_n^\perp) \text{ by continuity}\\ &= \lim_{n\to\infty}\langle w, w_n^\perp\rangle\\ &= \lim_{n\to\infty}(0) = 0,\end{aligned} \tag{5.25}$$

so $x \in W^\perp$. Therefore, $W^\perp$ is a closed subspace of H. □

Lemma 5.5.3. *(Dym and McKean [2, p. 29].) Let H be a Hilbert space. For any $v \in H$ and any orthonormal subset $\{e_n\}_{n=1}^\infty \subseteq H$, $\sum_{k=1}^\infty |\langle v, e_k\rangle|^2$ converges.*

Proof. Since $\sum_{k=1}^N |\langle v, e_k\rangle|^2 \le \|v\|$ by Bessel's inequality (Theorem 3.6.3), $\sum_{k=1}^\infty |\langle v, e_k\rangle|^2$ converges absolutely. □

Lemma 5.5.4. *(Griffel [3, p. 261, 262]). Let W be a closed subspace of a Hilbert space H and let $v \in H$. Then there exists in W an element w_0 such that $\|v - w_0\|$ is minimal, which means*

$$\|v - w_0\| \le \|v - w\| \text{ for all } w \in W. \tag{5.26}$$

Proof. Since

$$\{\|v - w\| \mid w \in W\}$$

is bounded below by 0, it has an infimum ℓ. Choose $S = \{w_n\}_{n=1}^\infty \subseteq W$ so that $\lim_{n\to\infty}\|v - w_n\| = \ell$. By Lemmas 4.5.5 and 4.5.6, there exists an orthonormal set $\{e_n\}_{n=1}^\infty = B$ in W such that span B = span S. In fact, by construction, we can arrange to have $\{w_1, \ldots, w_n\} \subseteq$ span $\{e_1, \ldots, e_n\}$. By lemmas 5.4.2 and 5.5.3, $\sum_{k=1}^\infty \langle v, e_k\rangle e_k$ converges. Let $w_0 = \sum_{k=1}^\infty \langle v, e_k\rangle e_k$. Since W is closed, $w_0 \in W$. Because each $w_n \in$ span $\{e_1, \ldots, e_n\}$, by Theorem 4.5.7, we have

$$\ell \le \left\| v - \sum_{k=1}^n \langle v, e_k\rangle e_k \right\| \le \|v - w_n\|. \tag{5.27}$$

Then, since $\lim_{n\to\infty}\|v - w_n\| = \ell$, $\lim_{n\to\infty}\|v - \sum_{k=1}^n \langle v, e_k\rangle e_k\| = \ell$. So

$$\|v - w_0\| = \left\| v - \sum_{k=1}^\infty \langle v, e_k\rangle e_k \right\| = \ell \tag{5.28}$$

because $\|.\|$ is continuous. □

Lemma 5.5.5. *(Reed and Simon [6, p. 41]). Let W be a subspace of a Hilbert space H, and let v be in H. If $w_0 \in W$ and $\|v - w_0\|$ is minimal (as in Lemma 5.5.4), then $v - w_0 \in W^\perp$.*

Proof. We want to show that for any $w \in W$, $\langle w, v - w_0 \rangle = 0$. Consider $\kappa : \mathbb{R} \to \mathbb{R}$ such that

$$\begin{aligned} \kappa(\epsilon) = \|v - w_0 + \epsilon w\|^2 &= \langle v - w_0 + \epsilon w, v - w_0 + \epsilon w \rangle \\ &= \|v - w_0\|^2 + 2\mathrm{Re}\langle \epsilon w, v - w_0 \rangle + \|\epsilon w\|^2 \\ &= \|v - w_0\|^2 + 2\epsilon \mathrm{Re}\langle w, v - w_0 \rangle + \epsilon^2 \|w\|^2 . \end{aligned} \tag{5.29}$$

$\kappa(\epsilon)$ is a differentiable function of ϵ. By assumption, $\kappa(\epsilon)$ has a minimum for $\epsilon = 0$, so $\kappa'(0) = 0$. But since

$$\kappa'(\epsilon) = 0 + 2\mathrm{Re}\langle w, v - w_0 \rangle + 2\epsilon \|w\|^2 , \tag{5.30}$$

$\kappa'(0) = 2\mathrm{Re}\langle w, v - w_0 \rangle$ which implies $0 = \mathrm{Re}\langle w, v - w_0 \rangle$. A similar argument for iw results to $\mathrm{Im}\langle w, v - w_0 \rangle = 0$. Therefore $\langle w, v - w_0 \rangle = 0$. □

Lemma 5.5.6. *(Reed and Simon [6, p. 41]). Let W be a subspace of a Hilbert space H. Then any $v \in H$ can be written in at most one way as $v = w + w^\perp$ where $w \in W$ and $w^\perp \in W^\perp$.*

Proof. Let

$$v = w + w^\perp = w' + w'^\perp \tag{5.31}$$

where $w, w' \in W$ and $w^\perp, w'^\perp \in W^\perp$. Then $w - w' = w'^\perp - w^\perp$. However, $w - w' \in W$ is orthogonal to $w'^\perp - w^\perp \in W^\perp$. Since the only vector orthogonal to itself is the zero vector, we have

$$w - w' = w'^\perp - w^\perp = 0, \tag{5.32}$$

which implies $w = w'$ and $w^\perp = w'^\perp$. □

Theorem 5.5.7. *(Reed and Simon [6, p. 42]). The Projection theorem: Let W be a closed subspace of a Hilbert space H, and let v be in H. Then there exists a unique vector $Pv \in W$ such that*

(1) $\|v - Pv\|$ is minimal (in the sense of Lemma 5.5.4).

(2) $v - Pv \in W^\perp$.

(3) We have $v = w + w^\perp$ with $w \in W$, $w^\perp \in W^\perp$ if and only if $w = Pv$, $w^\perp = v - Pv$.

Proof. Let $Pv = w_0$ from Lemma 5.5.4.

(1) By Lemma 5.5.4, $\|v - Pv\|$ is minimal.

(2) By Lemma 5.5.5, $v - Pv \in W^{\perp}$.

(3) By Lemma 5.5.6, $v = Pv + (v - Pv)$ is the unique decomposition of v of the form $v = w + w^{\perp}$ with $w \in W,\ w^{\perp} \in W^{\perp}$.

$\square$

Definition 5.5.8. (Dym and McKean [2, p. 28].) Let W be a closed subspace of a Hilbert space H. We say that $P : H \to W$ defined by Theorem 5.5.7, is the projection onto W.

Corollary 5.5.9. *(Reed and Simon [6, p. 42]). Let W be a closed subspace of a Hilbert space H, and let $P : H \to W$ be the projection onto W. Then for any $v \in H$,*

$$\|v\| \geq \|Pv\|. \tag{5.33}$$

Proof. Let W be a closed subspace of a Hilbert space H, and $W^{\perp}$ be a subspace of H. By theorem 5.5.7, for any $v \in H$ we have $v = w + w^{\perp}$, where $w = Pv \in W,\ w^{\perp} = v - Pv \in W^{\perp}$. Then

$$\begin{aligned} \|v\|^2 &= \langle w + w^{\perp}, w + w^{\perp} \rangle \\ &= \langle w, w \rangle + \langle w, w^{\perp} \rangle + \langle w^{\perp}, w \rangle + \langle w^{\perp}, w^{\perp} \rangle \\ &= \|w\|^2 + \|w^{\perp}\|^2. \end{aligned} \tag{5.34}$$

Therefore, $\|v\|^2 \geq \|w\|^2 = \|Pv\|^2$. $\square$

Corollary 5.5.10. *(Reed and Simon [6, p. 262]). Let W be a closed subspace of Hilbert space H, and let $P : H \to W$ be the projection onto W. Then for any $w \in W,\ Pw = w$.*

Proof. We observe that $w = w + 0$ with $w \in W$ and $0 \in W^{\perp}$. So by Theorem 5.5.7, $Pw = w$. $\square$

Theorem 5.5.11. *Let W be a closed subspace of Hilbert space H. Then the projection operator $P : H \to W$ is linear.*

Proof. Let $P : H \to W$ be the projection onto W. Suppose $u, v \in H$ and $\alpha \in \mathbb{C}$. Then

$$v = Pv + (v - Pv), \tag{5.35}$$

$$u = Pu + (u - Pu), \tag{5.36}$$

and

$$u + v = (Pu + Pv) + [(v - Pv) + (u - Pu)]. \tag{5.37}$$

Since $u + v \in H$, by part 3 of theorem 5.5.7 there exists a unique $w \in W,\ w^{\perp} \in W^{\perp}$ such that $u + v = w + w^{\perp}$, namely $w = P(u + v)$ and $w^{\perp} = (u + v) - P(u + v)$. So,

$$P(u + v) = Pu + Pv. \tag{5.38}$$

Also, since $\alpha u \in H$ and

$$\alpha u = \alpha Pu + \alpha(u - Pu), \tag{5.39}$$

by part 3 of theorem 5.5.7, there exists a unique $w \in W$ and $w^\perp \in W^\perp$ such that $\alpha u = w + w^\perp$, namely, $w = P(\alpha u)$ and $w^\perp = \alpha u - P(\alpha u)$. So

$$P(\alpha u) = \alpha Pu. \tag{5.40}$$

□

Theorem 5.5.12. *(Reed and Simon [6, p. 307].) Let W be a closed subspace of a Hilbert space H. Then W is a Hilbert space.*

Proof. Let W be a closed subspace of a Hilbert space H. We like to show that if $\{w_n\}_{n=1}^\infty \subseteq W$ is Cauchy, then $\lim_{n\to\infty} w_n$ exists and is in W. Because $\{w_n\}_{n=1}^\infty \subseteq H$, then $\lim_{n\to\infty} w_n = w \in H$ by definition of completeness (Definition 2.5.7). Also, since $\lim_{n\to\infty} w_n = w$ exists, w must be in W by definition of closed set (Definition 5.1.4.) Therefore, W is complete.

We want to show that there exists $\{w_n\}_{n=1}^\infty \subseteq W$ that is a countable dense subset (Definition 2.5.2) of W. Let $\{u_n\}_{n=1}^\infty$ be a countable dense subset of H and $w_n = Pu_n$ where $P : H \to W$. We want to show that $\{w_n\}_{n=1}^\infty$ is a dense subset of W. For every $w \in W$ and $\epsilon > 0$, since $\{u_n\}_{n=1}^\infty$ is dense in H then there exists $u_n \in H$ such that $\|u_n - w\| < \epsilon$. Then we have

$$\|w_n - w\| = \|Pu_n - Pw\| = \|P(u_n - w)\| \leq \|u_n - w\| < \epsilon. \tag{5.41}$$

Therefore, every closed subspace of a Hilbert space is a Hilbert space. □

5.6 $L^2(\mathbb{R}^n)$ and $L^2(X)$ are Hilbert spaces

Theorem 5.6.1. *$L^2(\mathbb{R}^n)$ is a Hilbert space.*

Sketch of proof. (See Reed and Simon [6, p. 50] for definition of tensor product $\otimes$.)

(1) Show that $L^2(\mathbb{R}^{k+1})$ is isomorphic to the inner product space $L^2(\mathbb{R}^k) \otimes L^2(\mathbb{R})$.

(2) By induction on n, $L^2(\mathbb{R}^n)$ is a Hilbert space because $L^2(\mathbb{R})$ is a Hilbert space.

□

Theorem 5.6.2. *Let X be a measurable subset of $\mathbb{R}^n$. Then the space $L^2(X)$ is isomorphic (as an inner product space) to a closed subspace W of $L^2(\mathbb{R}^n)$.*

Proof. Let X be a measurable subset of $\mathbb{R}^n$. We define

$$W = \{f \in L^2(\mathbb{R}^n) \mid f(x) = 0 \text{ for all } x \notin X\}, \tag{5.42}$$

and we want to show that

(1) W is a closed subspace of $L^2(\mathbb{R}^n)$.

(2) $L^2(X)$ is isomorphic to W.

W is a subspace of $L^2(\mathbb{R}^n)$ because for any $f, h \in W$, $\alpha \in \mathbb{C}$ and $x \notin X$,

$$(f+g)(x) = f(x) + h(x) = 0 + 0 = 0, \tag{5.43}$$

and

$$\alpha f(x) = \alpha\, 0 = 0. \tag{5.44}$$

We want to show that W and $L^2(X)$ are isomorphic. Define $\Phi : L^2(X) \to W$ by

$$(\Phi f)(x) = \begin{cases} f(x) & \text{if } x \in X, \\ 0 & \text{if } x \notin X. \end{cases}$$

Φ is a bijective function because $\Phi^{-1} : W \to L^2(X)$ is given by $(\Phi^{-1} f) = f\mid_x$ (the restriction of f to x). Moreover, for any $f \in L^2(X)$, $x \in X$,

$$\begin{aligned} \langle \Phi(f), \Phi(g) \rangle_W &= \int_{\mathbb{R}^n} \Phi(f)\overline{\Phi(g)} \\ &= \int_X \Phi(f)\overline{\Phi(g)} \\ &= \int_X f\overline{g} \\ &= \langle f, g \rangle_{L^2(X)}. \end{aligned} \tag{5.45}$$

Finally, we want to show that if $g \notin W$, then g is not a limit point of W. Let

$$S = \{x \in \mathbb{R}^n \backslash\ X \mid g(x) \neq 0\}. \tag{5.46}$$

Suppose $g \notin W$, which means $\mu(S) > 0$. It is enough to show that there exists a fixed positive constant q such that $d(f, g) > q$ for all $f \in W$, with q independent of f. Let

$$S_k = \{x \in \mathbb{R}^n \backslash X \mid |g(x)| \geq \frac{1}{k}\}. \tag{5.47}$$

Since $S = \cup_{k=1}^{\infty} S_k$, we must have $\mu(S_k) > 0$ for some $k \in \mathbb{N}$. For any $f \in W$,

$$\|f - g\|^2 = \int_{\mathbb{R}} |f-g|^2 \geq \int_{S_k} |f-g|^2 = \int_{S_k} |g|^2 \geq \int_{S_k} \frac{1}{k^2} = \frac{\mu(S_k)}{k^2} > 0. \tag{5.48}$$

So $d(f, g) \geq \dfrac{\mu(S_k)}{k^2} > 0$. Therefore, taking $q = \dfrac{\mu(S_k)}{k^2}$, g is not a limit point of W. $\square$

Corollary 5.6.3. *Let X be a measurable subset of $\mathbb{R}^n$. Then the space $L^2(X)$ is a Hilbert space.*

Proof. By theorem 5.6.2, $L^2(X)$ is isomorphic to a closed subspace of the Hilbert space $L^2(\mathbb{R}^n)$, so by theorem 5.5.12, $L^2(X)$ is a Hilbert space. $\square$

CHAPTER 6

COMPACT SELF-ADJOINT OPERATORS

6.1 Compact and self-adjoint operators

Definition 6.1.1. (Reed and Simon [6, p. 199].) A linear operator $K : V \to V$ on an inner product space V is compact if for every bounded subset (Definition 5.1.1) S of V, $K(S)$ is a relatively compact subset (Definition 5.1.8) of V.

Theorem 6.1.2. *(Reed and Simon [6, p. 199].) Let V be an inner product space. A linear operator $K : V \to V$ is compact if and only if for every bounded sequence $\{v_n\}_{n=1}^{\infty} \subset V$, $\{Kv_n\}_{n=1}^{\infty}$ has a subsequence $\{Kv_{n_k}\}_{k=1}^{\infty}$ that converges in V.*

Proof. 1 $\Rightarrow$ 2. Let a linear operator $K : V \to V$ on an inner product space V be compact. By definition 6.1.1, K maps any bounded sequence $\{v_n\}_{n=1}^{\infty} \subset V$ to a relatively compact set $\{Kv_n\}_{n=1}^{\infty}$ in V. So by theorem 5.1.9, there exists a subsequence $\{Kv_{n_k}\}_{k=1}^{\infty}$ which converges in V.

2 $\Rightarrow$ 1. Let $S \subset V$ be bounded, and consider a sequence $\{Kv_n\}_{n=1}^{\infty} \subseteq K(S)$ for any $v_n \in S$. Since S is bounded, $\{v_n\}_{n=1}^{\infty} \subseteq S$ is bounded. Thus, by assumption, $\{Kv_n\}_{n=1}^{\infty}$ has a convergent subsequence. So $K(S)$ is relatively compact by theorem 5.1.9. Therefore, by definition 6.1.1, K is compact. □

Theorem 6.1.3. *(Reed and Simon [6, p. 201].) If a linear operator $K : V \to V$ on an inner product space V is compact (Definition 6.1.1), then K is continuous.*

Proof. Let $K : V \to V$ be a compact linear operator (Definition 6.1.1), and let $B = \{v \in V \mid \|v\| = 1\}$. Then $\overline{K(B)}$ is compact, so $\overline{K(B)}$ is bounded (Theorem 5.1.10). Thus, there exists an $M \in \mathbb{N}$ such that $\|Kv\| \leq M$ for all $v \in B$. So by theorem 5.2.6, K is bounded. Therefore, by theorem 5.2.4, K is continuous. □

Lemma 6.1.4. *(Reed and Simon [6, p. 209].) Let $K : H \to H$ be a compact operator on a Hilbert space H, H_1 be a closed subspace of H, and $K(H_1) \subseteq H_1$. Then $K_1 : H_1 \to H_1$ where $K_1 = K|_{H_1}$, is compact.*

Proof. Suppose $\{v_n\}_{n=1}^{\infty}$ is bounded in $H_1 \subseteq H$. Since K is compact, $\{Kv_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{Kv_{n_k}\}_{k=1}^{\infty}$ in H.

We observe that $\{Kv_n\}_{n=1}^{\infty} \in H_1$ because $K(H_1) \subseteq H_1$, and $\lim_{k\to\infty} Kv_{n_k} \in H_1$ since H_1 is a closed subspace of H. Therefore, $\{Kv_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{Kv_{n_k}\}_{k=1}^{\infty} \in H_1$. By theorem 6.1.2, K_1 is compact. □

Theorem 6.1.5. *The identity operator* $id : \ell_2(\mathbb{Z}^+) \to \ell_2(\mathbb{Z}^+)$ *is bounded and continuous, but not compact.*

Proof. Suppose $id : \ell_2(\mathbb{Z}^+) \to \ell_2(\mathbb{Z}^+)$ is the identity operator and

$$S = \{v \in \ell_2(\mathbb{Z}^+) \mid \|v\| \leq 1\} \subseteq \ell_2(\mathbb{Z}^+). \tag{6.1}$$

Then S is bounded, but by theorem 5.3.4, $K(S) = S$ is not relatively compact as $\overline{k(S)} = \overline{S} = S$ is not compact. □

Definition 6.1.6. (Griffel [3, p. 222]). We say that a linear operator $A : V \to V$ on an inner product space V is self-adjoint if

$$\langle u, Av\rangle = \langle Au, v\rangle \text{ for all } u, v \in V.$$

6.2 Eigenvalues of self-adjoint operators

Definition 6.2.1. (Messer [5, p. 304].) Let $A : V \to V$ be a linear operator on an inner product space V. If $\lambda \in \mathbb{C}$ and $v \in V$ is a nonzero vector such that

$$Av = \lambda v, \tag{6.2}$$

then λ is an eigenvalue of A, and v is an eigenvector of A with eigenvalue λ.

Definition 6.2.2. Let $A : V \to V$ be a linear operator on an inner product space V, and let $\lambda \in \mathbb{C}$ be an eigenvalue of A. Then $E_A(\lambda) = \{v \in V \mid Av = \lambda v\}$ is the eigenspace of A associated with $\lambda \in \mathbb{C}$.

Theorem 6.2.3. *(Griffel [3, p. 228]). Let* $A : V \to V$ *be a self-adjoint operator on an inner product space* V*. Then*

(1) For any $v \in V$, $\langle v, Av\rangle \in \mathbb{R}$.

(2) Any eigenvalue (Definition 6.2.1) of A *is a real number.*

(3) Eigenvectors of A *with different eigenvalues are orthogonal.*

Proof. (1) If $A : V \to V$ is a self-adjoint operator on an inner product space V, then by Definition 3.1.1,

$$\overline{\langle v, Av\rangle} = \langle Av, v\rangle.$$

But by Definition 6.1.6,

$$\langle Av, v\rangle = \langle v, Av\rangle.$$

Therefore, $\langle v, Av\rangle \in \mathbb{R}$.

(2) If λ is an eigenvalue of A with eigenvector $v \in V$,

$$\langle Av, v \rangle = \langle \lambda v, v \rangle = \lambda \|v\|^2. \tag{6.3}$$

So since $\langle Av, v \rangle$ and $\|v\|^2$ are real, λ is real.

(3) If also $Aw = \mu w$ and $\lambda \neq \mu$, then since

$$\langle Av, w \rangle = \langle v, Aw \rangle, \tag{6.4}$$

we have

$$\langle Av, w \rangle - \langle v, Aw \rangle = 0. \tag{6.5}$$

So

$$\lambda \langle v, w \rangle - \mu \langle v, w \rangle = (\lambda - \mu) \langle v, w \rangle = 0. \tag{6.6}$$

Since $\lambda - \mu \neq 0,\ \langle v, w \rangle = 0$.

□

Lemma 6.2.4. *(Griffel [3, p. 224]). If a nonzero $\beta \in \mathbb{R}$ and $A : V \to V$ is a self-adjoint operator on an inner product space V, then for any $u \in V$,*

$$4\|Au\|^2 = \left\langle A\left(\beta u + \beta^{-1} Au\right), \beta u + \beta^{-1} Au \right\rangle - \left\langle A\left(\beta u - \beta^{-1} Au\right), \beta u - \beta^{-1} Au \right\rangle. \tag{6.7}$$

Proof. Consider a nonzero $\beta \in \mathbb{R}$, and let $A : V \to V$ be a bounded self-adjoint operator on an inner product space V. For any $u \in V$, we have

$$\begin{aligned}\left\langle A\left(\beta u \pm \beta^{-1} Au\right), \beta u \pm \beta^{-1} Au \right\rangle &= \left\langle \beta Au \pm \beta^{-1} A^2 u, \beta u \pm \beta^{-1} Au \right\rangle \\ &= \beta^2 \langle Au, u \rangle \pm \left\langle A^2 u, u \right\rangle \pm \langle Au, Au \rangle + \beta^{-2} \left\langle A^2 u, Au \right\rangle \\ &= \beta^2 \langle Au, u \rangle \pm 2 \langle Au, Au \rangle + \beta^{-2} \left\langle A^2 u, Au \right\rangle\end{aligned} \tag{6.8}$$

since A is self-adjoint. So

$$\begin{aligned}&\left\langle A\left(\beta u + \beta^{-1} Au\right), \beta u + \beta^{-1} Au \right\rangle - \left\langle A\left(\beta u - \beta^{-1} Au\right), \beta u - \beta^{-1} Au \right\rangle \\ &\quad = \beta^2 \langle Au, u \rangle + 2 \langle Au, Au \rangle + \beta^{-2} \left\langle A^2 u, Au \right\rangle \\ &\quad - \beta^2 \langle Au, u \rangle + 2 \langle Au, Au \rangle - \beta^{-2} \left\langle A^2 u, Au \right\rangle \\ &\quad = 4 \langle Au, Au \rangle \\ &\quad = 4\|Au\|^2.\end{aligned} \tag{6.9}$$

□

Lemma 6.2.5. *(Griffel [3, p. 74]). If V is an inner product space, then for all $u, v \in V$,*

$$\|u + v\|^2 + \|u - v\|^2 = 2\|u\|^2 + 2\|v\|^2. \tag{6.10}$$

Proof. Note that

$$\begin{aligned}\|u+v\|^2+\|u-v\|^2 &= \langle u+v,u+v\rangle + \langle u-v,u-v\rangle \\ &= \|u\|^2+\langle v,u\rangle+\langle u,v\rangle+\|v\|^2+\|u\|^2-\langle v,u\rangle-\langle u,v\rangle+\|v\|^2 \\ &= 2\|u\|^2+2\|v\|^2.\end{aligned} \tag{6.11}$$

□

Theorem 6.2.6. *If $A : V \to V$ is a bounded self-adjoint operator on an inner product space V, then*

$$\|A\| = \sup\{|\langle v, Av\rangle| \mid \|v\|=1\} \tag{6.12}$$

for any $v \in V$.

Proof. (Griffel [3, p. 224]). Let $A : V \to V$ be a bounded self-adjoint operator on an inner product space V, and $M = \sup\{|\langle v, Av\rangle| : \|v\|=1\}$. Then we have

$$\begin{aligned}|\langle v, Av\rangle| &\le \|v\|\|Av\| \text{ by Cauchy-Schwarz inequality} \\ &= \|Av\| \\ &\le \|A\|\|v\| \text{ by definition of } \|A\| \\ &= \|A\|.\end{aligned} \tag{6.13}$$

So $M \le \|A\|$.

Recall $\|A\| = \sup\{\|Av\| : \|v\|=1\}$ (Theorem 5.2.6). It therefore suffices to show that $\|Av\| \le M$ whenever $\|v\|=1$. If $\|Av\|=0$, then $\|Av\| \le M$, so assume $\|Av\| > 0$. Note that for any nonzero $x \in V$,

$$|\langle Ax, x\rangle| = \left| \|x\|^2 \left\langle A\left(\frac{x}{\|x\|}\right), \frac{x}{\|x\|}\right\rangle \right| \le M\|x\|^2, \tag{6.14}$$

and for $x = 0$, $|\langle Ax, x\rangle| = M\|x\|^2$. Then, for $\|v\|=1$, and $\beta \in \mathbb{R}$, Lemma 6.2.4 gives

$$\begin{aligned}4\|Av\|^2 &= \left\langle A\left(\beta v+\beta^{-1}Av\right), \beta v+\beta^{-1}Av\right\rangle - \left\langle A\left(\beta v-\beta^{-1}Av\right), \beta v-\beta^{-1}Av\right\rangle \\ &\le \left|\left\langle A\left(\beta v+\beta^{-1}Av\right), \beta v+\beta^{-1}Av\right\rangle\right| + \left|\left\langle A\left(\beta v-\beta^{-1}Av\right), \beta v-\beta^{-1}Av\right\rangle\right| \\ &\le M\left\|\beta v+\beta^{-1}Av\right\|^2 + M\left\|\beta v-\beta^{-1}Av\right\|^2 \text{ by (6.14)} \\ &= M\left(\left\|\beta v+\beta^{-1}Av\right\|^2 + \left\|\beta v-\beta^{-1}Av\right\|^2\right) \\ &= 2M\left(\|\beta v\|^2+\|\beta^{-1}Av\|^2\right) \text{ by Lemma 6.2.5} \\ &= 2M\left(\beta^2\|v\|^2+\beta^{-2}\|Av\|^2\right) = 2M(\beta^2+\beta^{-2}\|Av\|^2).\end{aligned} \tag{6.15}$$

Taking $\beta = \sqrt{\|Av\|}$,

$$4\left\|Av\right\|^2 \le 2M\left(\beta^2+\beta^{-2}\beta^4\right) = 4M\beta^2 = 4M\|Av\|. \tag{6.16}$$

So $\|Av\| \le M$. The theorem follows. □

6.3 The Hilbert-Schmidt theorem (Spectral theorem)

Theorem 6.3.1. *If $A : H \to H$ is a compact self-adjoint operator on a Hilbert space H, then there exists a unit eigenvector $v \in H$ such that $Av = \lambda v$, where $\lambda = \pm\|A\|$, and $|\langle v, Av\rangle| = \|A\| = \sup\{|\langle w, Aw\rangle| \mid \|w\| = 1\}$.*

Proof. (Griffel [3, p. 240]). Define $S = \{v \in H \mid \|v\| = 1\}$ and $f : S \to \mathbb{R}$ such that $f(v) = \langle v, Av\rangle$. By continuity of A and $\langle\ ,\ \rangle$, f is continuous. Since by theorem 6.2.6,

$$\|A\| = \sup_{v\in S} |f(v)|, \tag{6.17}$$

choose $\{v_n\}_{n=1}^{\infty} \in S$ such that $|f(v_n)| \to \|A\|$. Since $f(v_n) \in \mathbb{R}$, choose a subsequence $\{w_n\}_{n=1}^{\infty}$ of $\{v_n\}_{n=1}^{\infty}$ such that $f(w_n) \to \|A\|$ or $f(w_n) \to -\|A\|$, and let $\lambda = \lim_{n\to\infty} f(w_n)$. Since A is compact, $\{Aw_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{Ax_n\}_{n=1}^{\infty}$. Let $\lim_{n\to\infty} Ax_n = y$. We observe:

$$\begin{aligned}
\|Ax_n - \lambda x_n\|^2 &= \|Ax_n\|^2 - 2\lambda\langle x_n, Ax_n\rangle + \lambda^2\|x_n\|^2 \\
&\le \|A\|^2 - 2\lambda\langle x_n, Ax_n\rangle + \lambda^2\|x_n\|^2 \\
&\le \|A\|^2 - 2\lambda f(x_n) + \lambda^2 \\
&= 2\lambda^2 - 2\lambda f(x_n) \to 0
\end{aligned} \tag{6.18}$$

because $\{x_n\}$ is a subsequence of $\{w_n\}$ and $f(w_n) \to \lambda$. So $\lim_{n\to\infty}(Ax_n - \lambda x_n) = 0$. Since $y - \lambda x_n = (Ax_n - \lambda x_n) + (y - Ax_n) \to 0$, $\lambda x_n \to y$. If $\lambda = 0$, $\|A\| = 0$, and $Av = 0v$ for any unit $v \in H$. If $\lambda \neq 0$, choose $v = \frac{1}{\lambda}y$, so $x_n \to v$. Thus, $Av = A(\lim_{n\to\infty} x_n)$. Since A is continuous,

$$Av = \lim_{n\to\infty} Ax_n = \lambda v. \tag{6.19}$$

Finally, note that $\|v\| = \|\lim_{n\to\infty} x_n\| = \lim_{n\to\infty}\|x_n\| = 1$ by continuity of $\|.\|$, and

$$|\langle v, Av\rangle| = |f(v)| = \left|f\left(\lim_{n\to\infty} x_n\right)\right| = \left|\lim_{n\to\infty} f(x_n)\right| = \|A\| \tag{6.20}$$

by continuity of f. $\square$

Theorem 6.3.2. *(Griffel [3, p. 243]). Let $A : H \to H$ be a compact self-adjoint operator on an infinite-dimensional Hilbert space H. Then there exists an orthonormal set of eigenvectors $\{e_n\}_{n=1}^{\infty}$ with corresponding eigenvalues λ_n such that*

(1) $\{e_n\}_{n=1}^{\infty}$ is a basis for H.

(2) $\lim_{n\to\infty} \lambda_n = 0$.

(3) For any $\lambda_n \neq 0$, the dimension of $E_A(\lambda_n)$ is finite. (I.e., each nonzero λ_n has finite multiplicity.)

Proof. (Griffel [3, p. 243]). Let $A : H \to H$ be a compact self-adjoint operator on an infinite-dimensional Hilbert space H, and let $H_0 = H$, $A_0 = A$. By theorem 6.2.6, choose a unit eigenvector e_1 such that $Ae_1 = \lambda_1 e_1$, $\lambda_1 = \pm\|A\|$, and $|\langle e_1, Ae_1\rangle|$ is maximal among unit vectors. By lemma 5.5.7, $H_1 = (\text{ span}\{e_1\})^{\perp}$ is a closed subspace of H, so by theorem 5.5.12, H is a Hilbert space. For $v \in H_1$,

$$\begin{aligned}\langle Av, e_1\rangle &= \langle v, Ae_1\rangle \\ &= \langle v, \lambda_1 e_1\rangle \\ &= \overline{\lambda_1}\langle v, e_1\rangle \\ &= 0.\end{aligned} \tag{6.21}$$

So $A(H_1) \subseteq H_1$. Thus, by lemma 6.1.4, $A_1 = A\,|_{H_1}$ is a compact self-adjoint operator $A_1 : H_1 \to H_1$, and

$$\begin{aligned}\|A_1\| &= \sup\{|\langle w, Aw\rangle| \mid \|w\| = 1,\ w \in H_1\} \\ &\le \sup\{|\langle w, Aw\rangle| \mid \|w\| = 1,\ w \in H\} = \|A_0\| = |\lambda_1|\,.\end{aligned} \tag{6.22}$$

For $k \geq 1$, consider a compact self-adjoint operator $A_k : H_k \to H_k$. Suppose an orthonormal set of eigenvectors $\{e_i\}_{i=1}^{k}$ with eigenvalues λ_i such that

$$\|A_0\| = |\lambda_1| \ge \|A_1\| = |\lambda_2| \ge \|A_2\| = |\lambda_3| \ge, \ldots, \|A_{k-1}\| = |\lambda_k| \ge \|A_k\| \tag{6.23}$$

has already been chosen. Let $e_{k+1} \in H_k$ be an eigenvector such that $\|e_{k+1}\| = 1$, $A_k e_{k+1} = \lambda_{k+1} e_{k+1}$, $\lambda_{k+1} = \pm\|A_k\|$ and $|\langle e_{k+1}, A_k e_{k+1}\rangle|$ is maximal by theorem 6.2.6. Let $H_{k+1} = (\text{ span } \{e_{k+1}\})^{\perp} \cap H_k$, which is a closed subspace of H_k, by lemma 5.5.2, and therefore, a Hilbert space, by theorem 5.5.12. For all $v \in H_{k+1}$, $\langle v, e_{k+1}\rangle = 0$. Since A_k is self-adjoint,

$$\begin{aligned}\langle A_k v, e_{k+1}\rangle &= \langle v, A_k e_{k+1}\rangle \\ &= \langle v, \lambda_{k+1} e_{k+1}\rangle \\ &= \overline{\lambda_{k+1}}\langle v, e_{k+1}\rangle \\ &= 0.\end{aligned} \tag{6.24}$$

Thus, $A_k(H_{k+1}) \subseteq H_{k+1}$. So we can define $A_{k+1} : H_{k+1} \to H_{k+1}$ and as above, $\|A_{k+1}\| \le \|A_k\| = |\lambda_{k+1}|$.

Letting k go to ∞, we see that

(1) $\{e_n\}_{n=1}^{\infty}$ is an orthonormal subset of H.

(2) $Ae_n = \lambda_n e_n$.

(3) $H_k = (\text{ span } \{e_1, e_2, \ldots, e_k\})^{\perp}$ is a closed subspace of H.

(4) $A\,|_{H_k}$ is $A_k : H_k \to H_k$.

(5) $\|A_0\| = |\lambda_1| \geq \|A_1\| = |\lambda_2| \geq \|A_2\| = |\lambda_3| \geq \ldots.$

Now, since $\{e_n\}_{n=1}^{\infty}$ is a bounded subset of H and A is compact, $\{Ae_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{Ae_{n_k}\}_{n=1}^{\infty}$. Let $v = \lim_{k\to\infty} Ae_{n_k}$. By lemma 5.5.3 and (1), $\sum |\langle Av, e_{n_k}\rangle|^2$ converges, so $|\langle Av, e_{n_k}\rangle| \to 0$. Therefore,

$$\begin{aligned} \langle v, v\rangle &= \langle \lim_{k\to\infty} Ae_{n_k}, v\rangle \\ &= \lim_{k\to\infty} \langle Ae_{n_k}, v\rangle \text{ by continuity of } \langle\ ,\ \rangle \\ &= \lim_{k\to\infty} \langle e_{n_k}, Av\rangle \text{ since } A \text{ is self-adjoint} \\ &= 0. \end{aligned} \tag{6.25}$$

So $v = \lim_{k\to\infty} Ae_{n_k} = 0$, which means by (2), $|\lambda_{n_k}| = \|Ae_{n_k}\| \to 0$. Therefore, since $|\lambda_{n_k}| \to 0$, and $|\lambda_n|$ is decreasing by (5), then $|\lambda_n| \to 0$ (the second part of theorem). It also follows that each nonzero λ_n has finite multiplicity.

Let $H_\infty = (\text{span } \{e_n\}_{n=1}^{\infty})^{\perp} = \bigcap_{k=1}^{\infty} H_k$ by (3). H_∞ is the intersection of closed subspaces, so H_∞ is also closed and therefore is a Hilbert space by theorem 5.5.12. Furthermore, suppose $v \in (H_\infty) = \bigcap_{k=1}^{\infty} H_k$. So $v \in H_k$ for all k, which means that $Av \in H_\infty$ by (4). Therefore, $A\ |_{H_\infty} = A_\infty : H_\infty \to H_\infty$ is well defined.

Furthermore, suppose $v \in H_\infty$, $\|v\| = 1$. Since $v \in H_k$, by (5),

$$\|A_\infty v\| = \|A_k v\| \leq \|A_k\| = |\lambda_{k+1}| \text{ for all } k. \tag{6.26}$$

Since $|\lambda_{k+1}| \to 0$ as $k \to \infty$, then $\|A_\infty v\| = 0$. This means $\|A_\infty v\| = 0$ for all $v \in H_\infty$. It follows that $H_\infty \subseteq$ null space A, and any non-zero vector in H_∞ is an eigenvector with eigenvalue 0.

So we choose an orthonormal basis $\{u_n\}_{n=1}^{\infty}$ for H_∞. Let $\{e_n\}_{n=1}^{\infty} \cup \{u_n\}_{n=1}^{\infty} = S$. Then S is orthonormal, and S spans H, by part (1) of theorem 5.5.7. The theorem follows. $\square$

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